Phase Space

Reconstruction
Brief Recap: Need for Nonlinear Methods

Lorenz oscillator

\[
\begin{align*}
\dot{x} &= 10(y - x) \\
\dot{y} &= x(28 - z) - y \\
\dot{z} &= xy - \frac{8}{3}z
\end{align*}
\]

AR(1) process measured with nonlinearity

\[
\begin{align*}
y_t &= 0.8y_{t-1} + \epsilon_t \\
x_t &= (1 + y_t)^2
\end{align*}
\]

from: G. Ansmann
Brief Recap: Need for Nonlinear Methods

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skewness \approx 0.004; kurtosis \approx -0.71

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skewness \approx 2.6; kurtosis \approx 9.7

from: G. Ansmann

indication for nonlinearity?
Brief Recap: Need for Nonlinear Methods

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Phase Space

Fundamentals of Analyzing Biomedical Signals

from: G. Ansmann
Brief Recap: Need for Nonlinear Methods

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Brief Recap: Need for Nonlinear Methods

When faced with time series from nonlinear systems, linear methods

- fail to detect the dynamics / structure in the data
- do not tell much about the dynamics
- cannot distinguish chaos from noise

→ Structure can be seen in attractors.
Brief Recap: Attractor

states of the dynamics for $t \to \infty$

type of dynamics can be deduced from topology of attractor:

- point $\to$ fixed-point dynamics
- limit cycle $\to$ periodic dynamics
- torus $\to$ quasiperiodic dynamics
- strange attractor $\to$ chaos

attractor reflects further central properties of dynamics.
Strange Attractors
Need for Phase-Space Reconstruction

Directly observing the phase space / attractor requires access to all the system’s dynamical variables.

But:
- often, only one dynamical variable accessible (or a time series thereof)
- dimension of phase space is often unknown

Can we obtain from a single time series a set that preserves important properties of the attractor?
Phase-Space Reconstruction

good and bad

- Actual attractor
- Perfect reconstruction
- Perfect reconstruction
- Preserves structure, practically bad
- Preserves structure, practically bad
- 2D: structure not preserved
- 3D: structure may be preserved
- (Phase as real number) structure not preserved
- (Forth and back on the same line) structure not preserved

from: G. Ansmann
Phase-Space Reconstruction

original attractor
→ a $d$-manifold $\mathcal{A} \subset \mathbb{R}^d$

measurement and reconstruction
→ a map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$

structure-preserving reconstruction
→ topology-preserving map → an embedding

**embedding**
a map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$ is called an embedding, if:

- $\nabla \phi$ has full rank
- $\phi$ is a diffeomorphism:
  - $\phi$ is differentiable
  - $\phi^{-1}$ exists and is differentiable
Phase-Space Reconstruction

**Strong Whitney embedding theorem**

For $m = 2d$, there exists a map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$ that is an embedding.

Problem: $\phi$ usually unknown

**Weak Whitney embedding theorem**

For $m > 2d+1$, almost every continuously differentiable ($C^1$) map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$ is an embedding.

Problems:
- Often, we do not have $m$ independent observables
  (redundant observables are one of the reasons for “almost every”)
- We do not know $d$
Phase-Space Reconstruction

Idea:

- given time series \( v: v_1, v_2, \ldots, v_N \) of some system observable \( x \)
- derivatives (first, second, third, \( \ldots \)) are not fully redundant.
- approximate derivatives with difference quotients:

\[
\dot{v}_i = v_{i+1} - v_i \\
\ddot{v}_i = v_{i+2} - 2v_{i+1} + v_i \\
\text{etc.}
\]

\( \sim v_i, v_{i+1}, \ldots \) are not fully redundant

\( \sim \) inverse Taylor expansion
Phase-Space Reconstruction

Takens’ Theorem:

- let $\mathcal{A} =: \{x_1, x_2, \ldots, x_N\}$ with the index indicating time
- let $h : \mathcal{A} \rightarrow \mathbb{R}$ denote the measurement function that maps the system observable $\mathbf{x}$ to the time series $\mathbf{v}$

If $m > 2d+1$,

$$\phi_{h,\tau} := (v_i, v_{i-\tau}, \ldots, v_{i-(m-1)\tau})$$

is an embedding for almost all dynamics, embedding delays $\tau$ and measurement functions $h$. $m$ denotes the embedding dimension.
Phase-Space Reconstruction

**Takens’ Theorem and applications:**

- given time series \( \mathbf{v} : v_1, v_2, ..., v_N \) of some system observable \( \mathbf{x} \)
- consider \( m \)-dimensional states (mapped from the attractor to the time series):

  \[
  \left( v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau} \right)^T
  \]

- for a proper embedding dimension \( m \) and embedding delay \( \tau \), these states make up a topologically equivalent reconstruction of the attractor.
Phase-Space Reconstruction

Example: Lorenz attractor

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Delay-Embeddings

Example: brain dynamics

EEG (awake state)
Phase-Space Reconstruction

Example: brain dynamics

EEG (epilepsy patient)
Phase-Space Reconstruction

Example: brain dynamics

EEG (epileptic seizure)
Dynamical Invariants

Important characteristics of the dynamics are invariant under the embedding transformation:

• Lyapunov exponents
• dimensions
• entropy
• ...

Phase-Space Reconstruction

Identifying embedding parameter

element: Lorenz attractor

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Identifying embedding parameter
delay
example: Lorenz attractor

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*Identifying embedding parameter*

Example: Lorenz attractor

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**Identifying embedding parameter**

delay

easy example: Lorenz attractor

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Identifying embedding parameter

example: Lorenz attractor

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**Identifying embedding parameter**

requirement for an embedding:
\((v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau})\) not fully redundant

→ aforementioned theorems: almost every \(\tau\) yields an embedding:

requirements for a **good embedding**:

- minimum redundancy of \((v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau})\)
  (to unfold the attractor)
- reasonably small \(\tau\)
  (to avoid folding the attractor onto itself)

(compare to: linear independence vs. orthogonality)
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**Identifying embedding parameter**

using zeros of the autocorrelation

**Idea:**

if autocorrelation = 0 for some delay $\Delta \Rightarrow v_t$ and $v_{t+\Delta}$ are **linearly** independent on average

$\rightarrow$ choose the first zero of the autocorrelation $\Delta$ as embedding delay
Phase-Space Reconstruction

Identifying embedding parameter

using the first minimum of mutual information $I$

**Idea:** if common information for some delay $\Delta$ is minimum $\Rightarrow$

$v_t$ and $v_{t-\Delta}$ are independent on average (also includes nonlinear relationships)

$I(M_1, M_2) = H(M_1) - H(M_1 | M_2) = H(M_1) + H(M_2) - H(M_1, M_2)$

where $M_1$ and $M_2$ denote measurements at times $t$ and $t-\Delta$, and

$H = - \sum_i p_i \log p_i$ is the Shannon entropy

→ choose the first minimum of the mutual information $\Delta$ as embedding delay

Phase-Space Reconstruction

**Identifying embedding parameter**

first minimum of mutual information $I$

---

Phase-Space Reconstruction

*Identifying embedding parameter*

- zeros of the autocorrelation function
- minima of the mutual information
- many more

No method is perfect or commonly agreed upon.

Practically:

- try at least two methods
- judge by further analysis
- alternative for $m \leq 3$: visually inspect the attractor
- keep embedding window $(m - 1)\tau$ (time span in an embedded vector) constant
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Identifying embedding parameter

Embedding theorems define “sufficient” embedding dimension $m$

Problem: dimension of system under study usually unknown

Choosing $m$ overly high may hamper further analyses

(impact of noise, finite number of data points, computational complexity)

→ Need other ways to determine a good $m$
Phase-Space Reconstruction

Identifying embedding parameter
dimension

Linear Dependence

idea:
• $m$ is higher than necessary
  ⇒ attractor only covers a subspace of reconstruction space
    (e.g., circle in $m = 3$)
• check whether embedded vectors have full rank.

difficulties:
• noise acts in all directions
• assumes linear dependence → dependence may be nonlinear
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**Identifying embedding parameter**

*Asymptotic Invariants*

idea:

- $m$ too small $\Rightarrow$ wrong dynamical invariants (in general)
- $m$ sufficient $\Rightarrow$ correct dynamical invariants

$\rightarrow$ increase $m$ until dynamical invariants converge

**difficulties:**

- criterion for convergence under real conditions (noise, finite number of data points, …)
- *wrong (in general)* is unpredictable
Phase-Space Reconstruction

Identifying embedding parameter dimension

False Nearest Neighbors

idea:

• $m$ too small $\Rightarrow$ trajectories intersect

$\Rightarrow$ points close in reconstruction space that aren’t close in actual phase space (false nearest neighbors)

$\Rightarrow$ increase $m$ until false nearest neighbors vanish.

Phase-Space Reconstruction

**Identifying embedding parameter**

**False Nearest Neighbors**

practically:

- choose threshold $\varepsilon$ for nearest neighbors
- $NN(m)$: number of pairs of points in $m$-dimensional reconstruction space that are closer than $\varepsilon$
- $NN(m + 1) < NN(m)$
  $\Rightarrow$ at least $NN(m) - NN(m + 1)$ false nearest neighbors in the $m$-dimensional reconstruction.

**Difficulties:**
number of true nearest neighbors large and fluctuating (noise).
Phase-Space Reconstruction

**Summary**

delay embedding allows to reconstruct attractor from single observable

- parameters $m$ and $\tau$ have to be carefully chosen
- reconstructed phase space may be used for:
  - understanding
  - prediction
  - modelling
  - …

- characteristics preserved by reconstruction:
  - dimensions, Lyapunov exponents, entropy, …
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Extensions

- multivariate time series
- different embedding delays for each component
- state-dependent embedding delays