Phase Space

Reconstruction

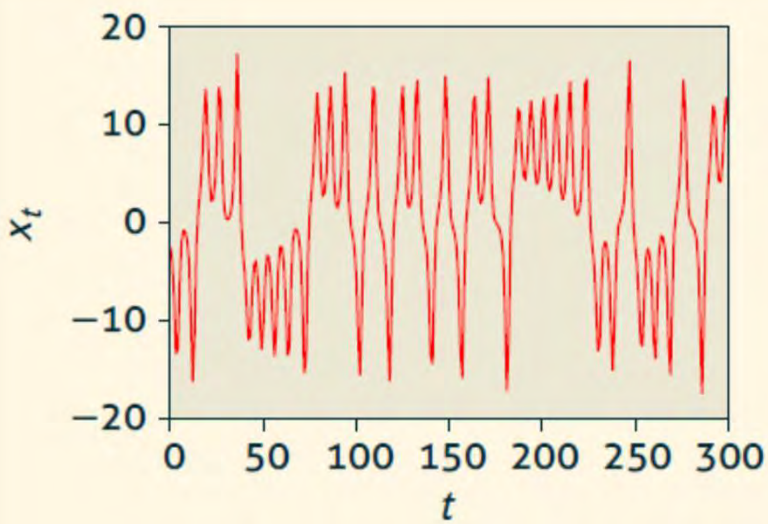
Brief Recap: Need for Nonlinear Methods

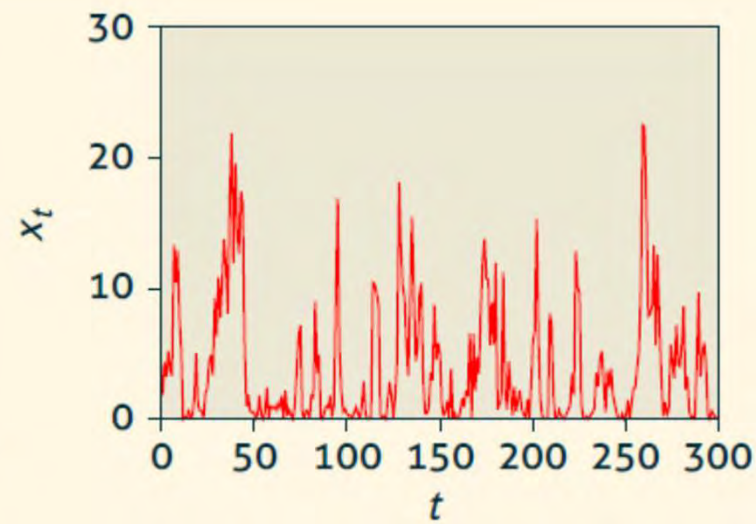
Lorenz oscillator

$$\begin{aligned} \dot{x} &= 10(y - x) \\ \dot{y} &= x(28 - z) - y \\ \dot{z} &= xy - \frac{8}{3}z \end{aligned}$$

AR(1) process measured with nonlinearity

$$\begin{aligned} y_t &= 0.8 y_{t-1} + \epsilon_t \\ x_t &= (1 + y_t)^2 \end{aligned}$$





from: G. Ansmann

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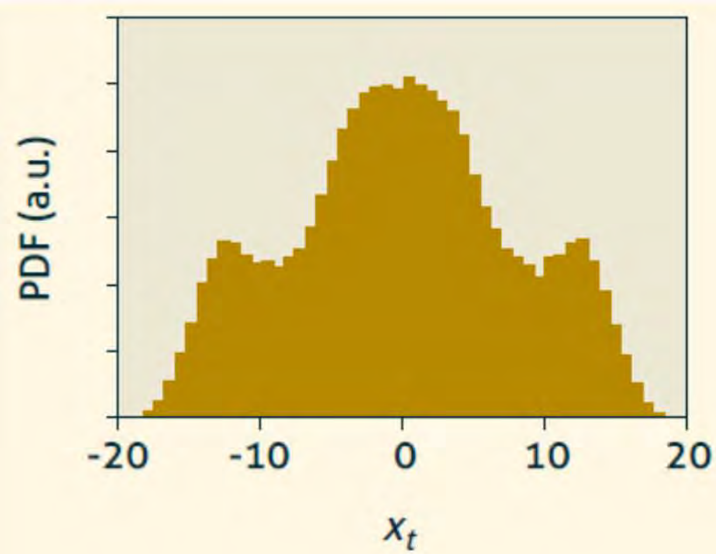
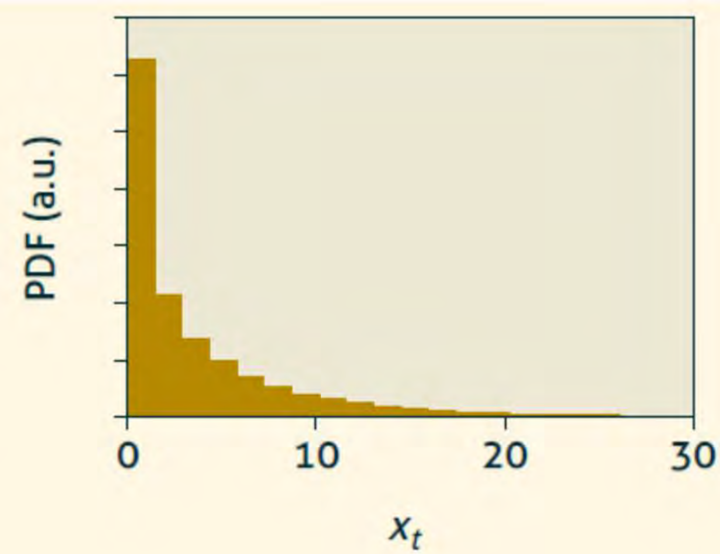
skewness ≈ 0.004; kurtosis ≈ -0.71

skewness ≈ 2.6; kurtosis ≈ 9.7





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indication for nonlinearity?

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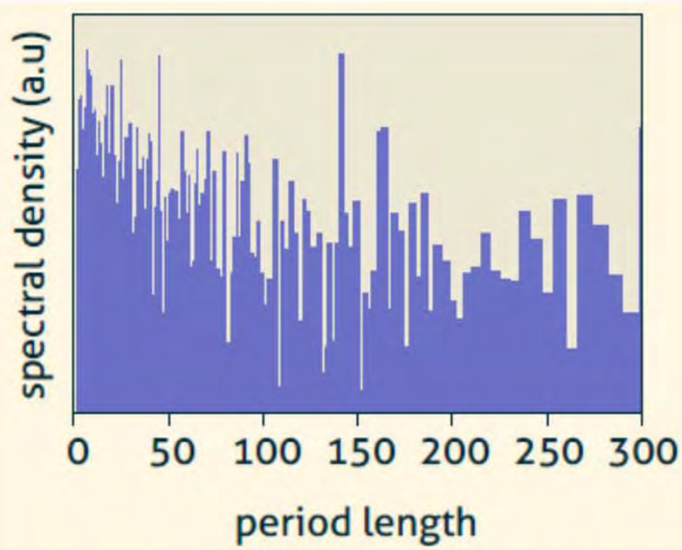

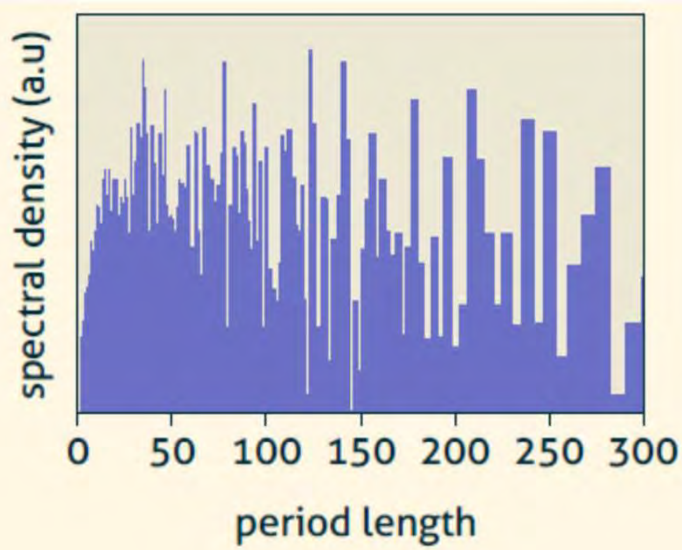

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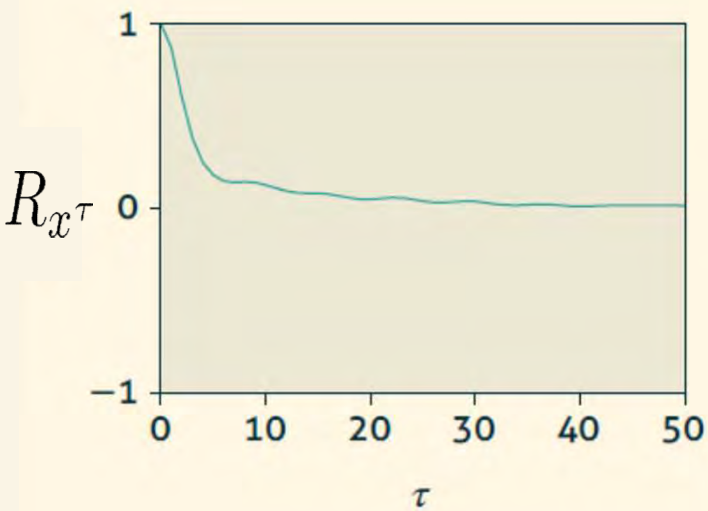

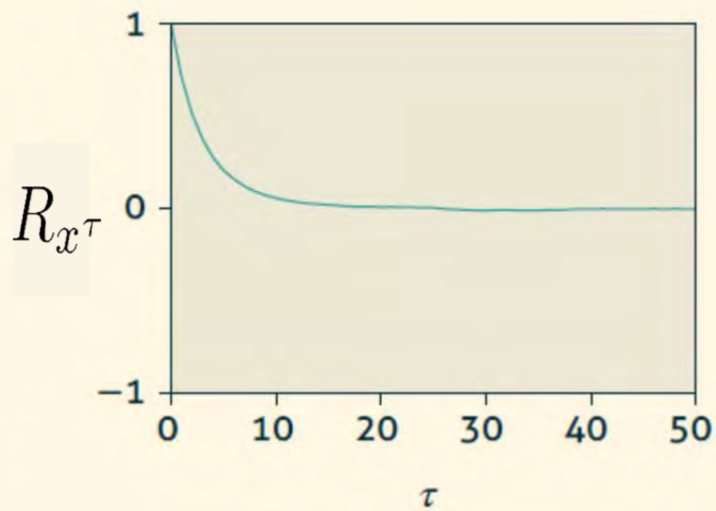

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Brief Recap: Need for Nonlinear Methods

When faced with time series from nonlinear systems, linear methods

- fail to detect the dynamics / structure in the data
- do not tell much about the dynamics
- cannot distinguish chaos from noise

→ Structure can be seen in attractors.

Brief Recap: Attractor

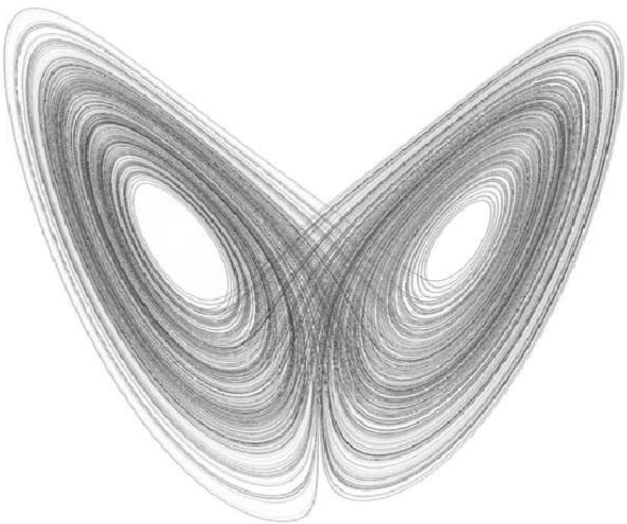
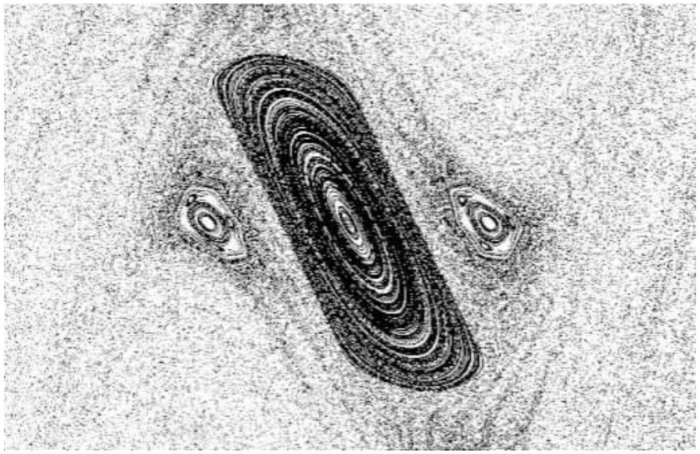
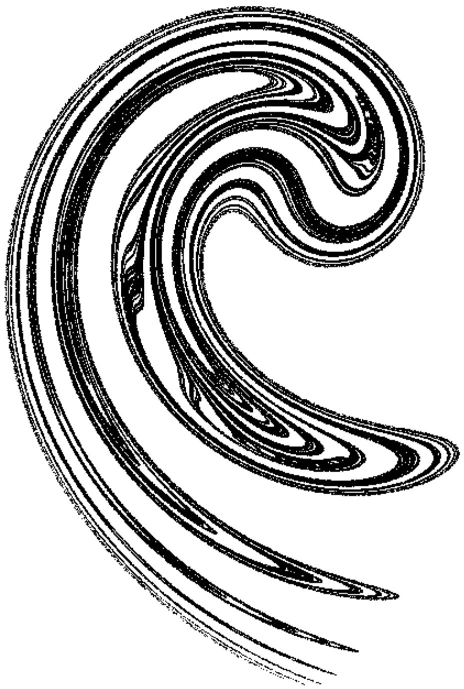
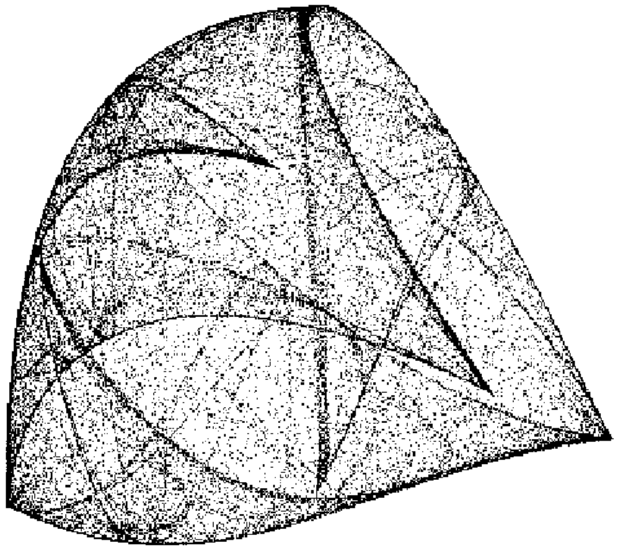
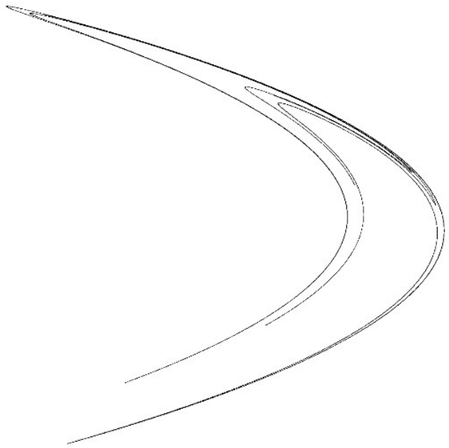
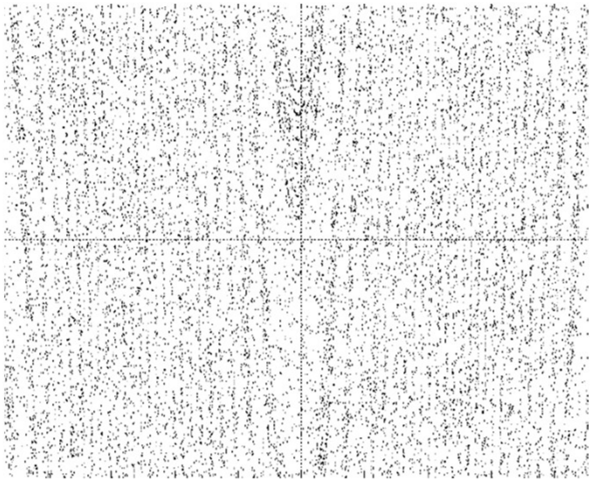
states of the dynamics for $t \rightarrow \infty$

type of dynamics can be deduced from topology of attractor:

- point → fixed-point dynamics
- limit cycle → periodic dynamics
- torus → quasiperiodic dynamics
- strange attractor → chaos

attractor reflects further central properties of dynamics.

Strange Attractors

Need for Phase-Space Reconstruction

Directly observing the phase space / attractor requires access to all the system's dynamical variables

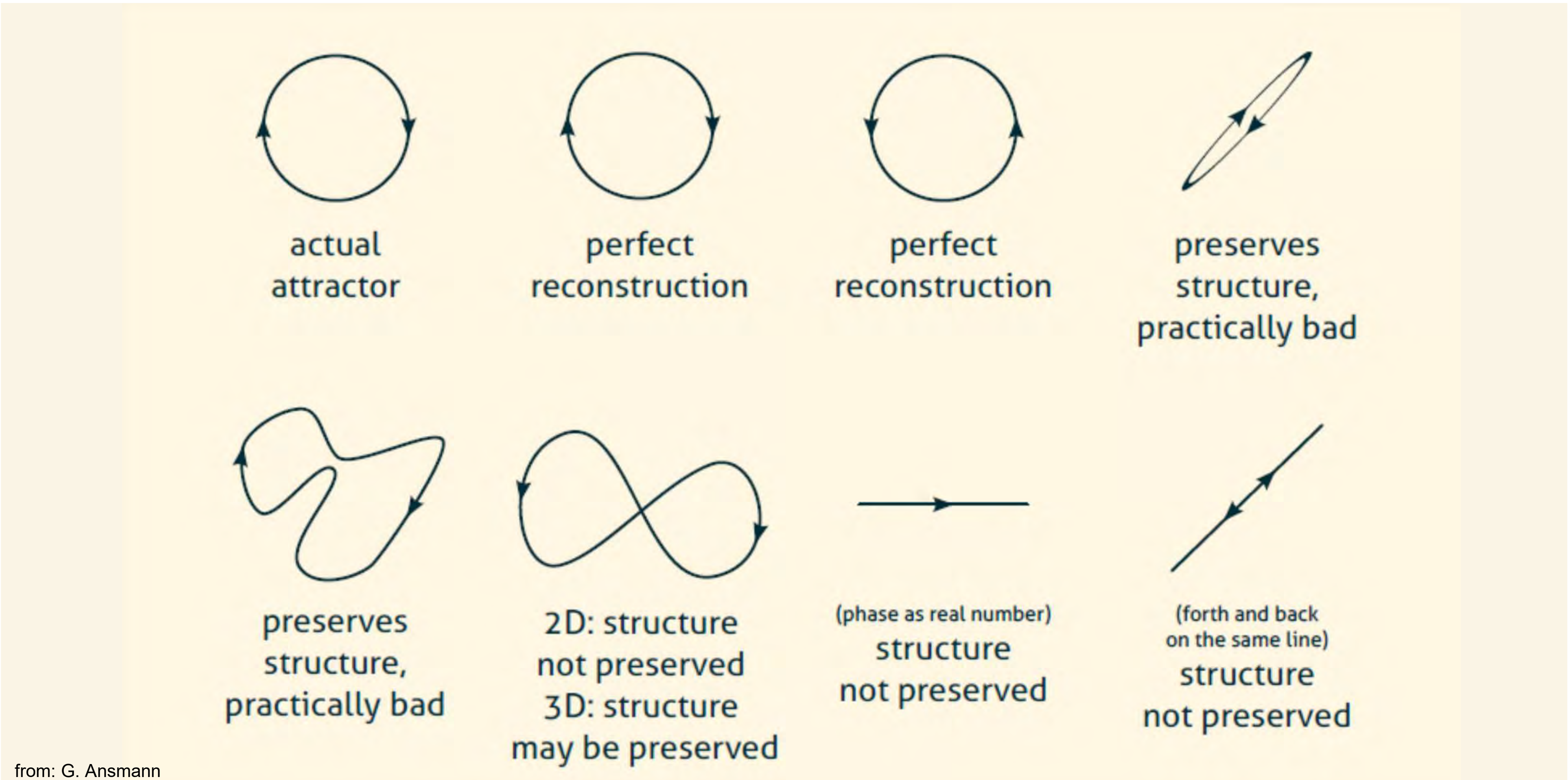
But:

- often, only one dynamical variable accessible (or a time series thereof)

- dimension of phase space is often unknown

Can we obtain from a single time series a set that preserves important properties of the attractor?

Phase-Space Reconstruction good and bad



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Phase-Space Reconstruction Topology

original attractor

$\longrightarrow$ a *d*-manifold $\mathcal{A} \subset \mathbb{R}^d$

measurement and reconstruction

$\longrightarrow$ a map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$

structure-preserving reconstruction

$\longrightarrow$ topology-preserving map $\longrightarrow$ an embedding

embedding

a map $\phi : \mathcal{A} \rightarrow \mathbb{R}^m$ is called an *embedding*, if:

- $\nabla \phi$ has full rank
- ϕ is a diffeomorphism:

  ϕ is differentiable

  ϕ^{-1} exists and is differentiable

Phase-Space Reconstruction **Embeddings**

Strong Whitney embedding theorem

For $m = 2d$, there exists a map $\phi : \mathcal{A} \to \mathbb{R}^m$ that is an embedding.

Problem: ϕ usually unknown

Weak Whitney embedding theorem

For $m > 2d+1$, almost every continuously differentiable (C^1) map $\phi : \mathcal{A} \to \mathbb{R}^m$ is an embedding.

Problems:

- Often, we do not have m independent observables (redundant observables are one of the reasons for "almost every")
- We do not know d

Phase-Space Reconstruction Delay-Embeddings

Idea:

- given time series $\boldsymbol{v}$: $v_1, v_2, \ldots, v_N$ of some system observable $\mathbf{x}$
- derivatives (first, second, third, …) are not fully redundant.
- approximate derivatives with difference quotients:

$$\begin{aligned} \dot{v}_i &= v_{i+1} - v_i \\ \ddot{v}_i &= v_{i+2} - 2v_{i+1} + v_i \end{aligned}$$

etc.

↝ v_i, v_{i+1}, … are not fully redundant

↝ inverse Taylor expansion

Phase-Space Reconstruction — Delay-Embeddings

Takens' Theorem:


Floris Takens

- let $\mathcal{A} =: \{x_1, x_2, \ldots, x_N\}$ with the index indicating time
- let $h : \mathcal{A} \rightarrow \mathbb{R}$ denote the measurement function that maps the system observable $\mathbf{x}$ to the time series v

If $m > 2d+1$,

$$\phi_{h,\tau} := \left(v_i, v_{i-\tau}, \ldots, v_{i-(m-1)\tau}\right)$$

is an embedding for almost all dynamics, *embedding delays* τ and measurement functions h. m denotes the *embedding dimension.*

Phase-Space Reconstruction Delay-Embeddings

Takens' Theorem and applications:

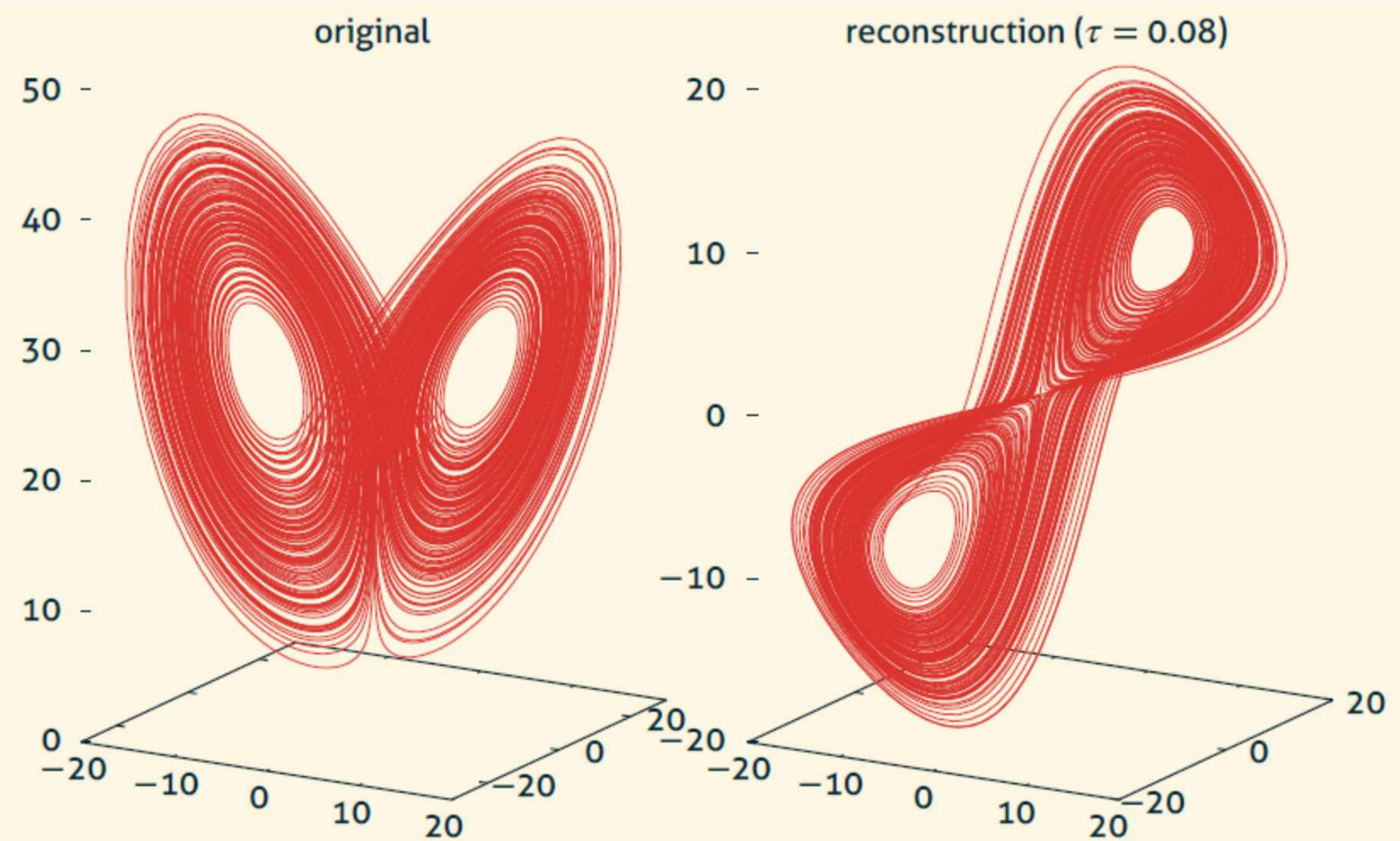
- given time series $\boldsymbol{v}$: $v_1, v_2, \ldots, v_N$ of some system observable $\mathbf{x}$
- consider *m*-dimensional states (mapped from the attractor to the time series:

$$\left(v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau}\right)^T$$

- for a proper embedding dimension *m* and embedding delay τ, these states make up a topologically equivalent reconstruction of the attractor.

Phase-Space Reconstruction Delay-Embeddings

Example: Lorenz attractor



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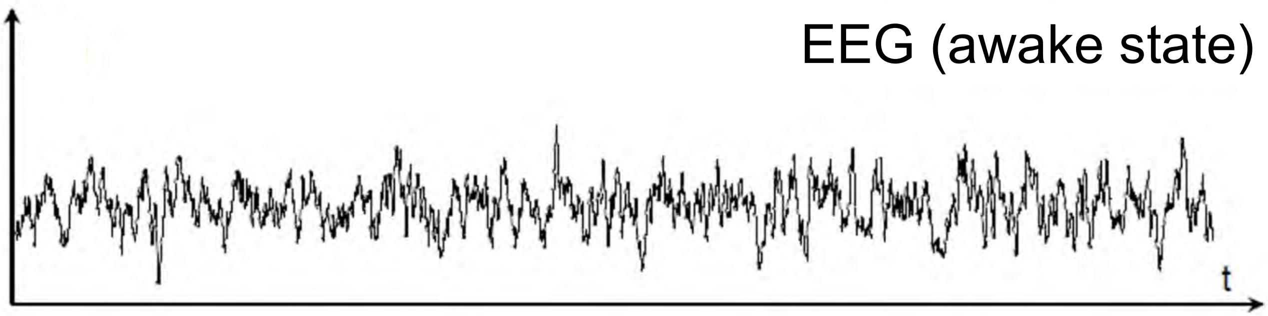
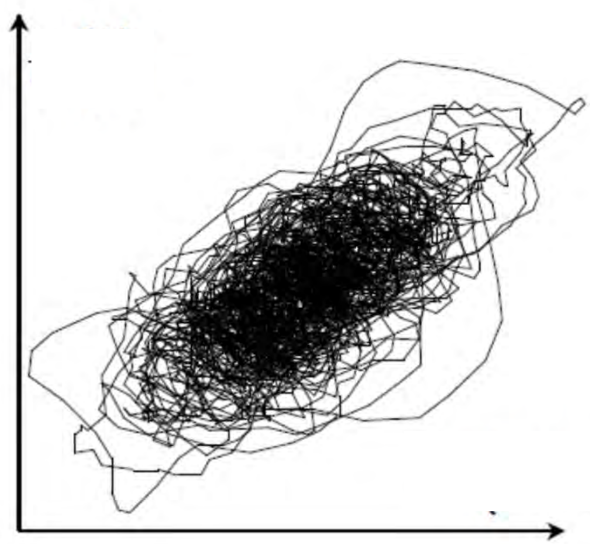
Phase-Space Reconstruction Delay-Embeddings

Example: brain dynamics

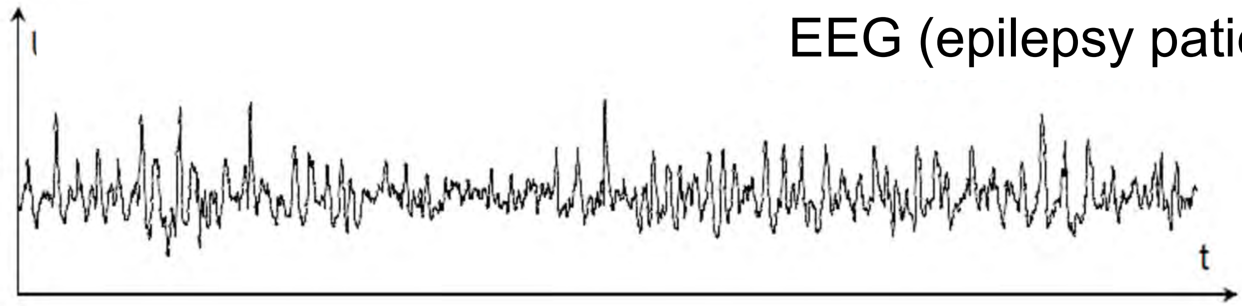




Phase-Space Reconstruction Delay-Embeddings

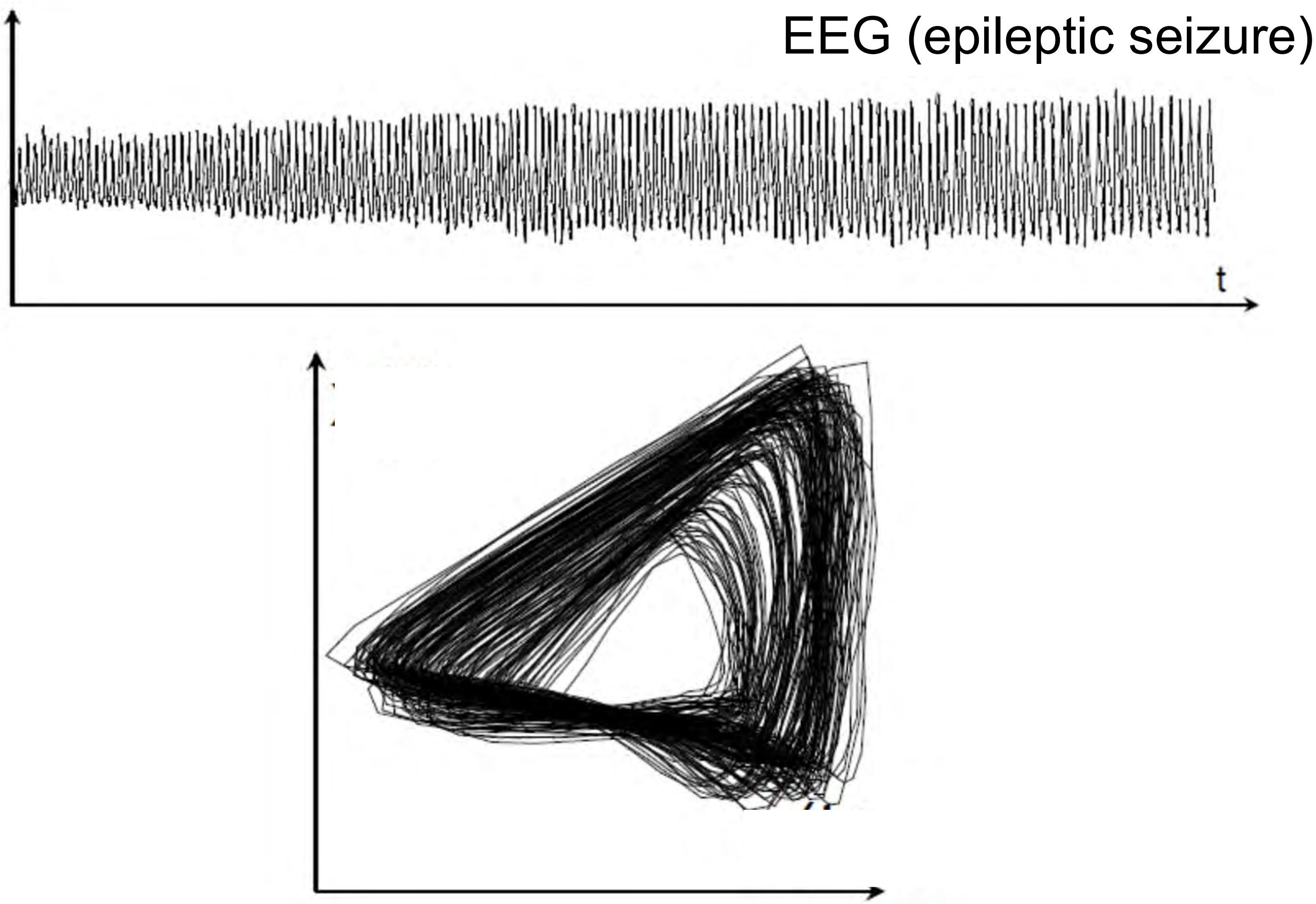
Example: brain dynamics

EEG (epilepsy patient)



Phase-Space Reconstruction Delay-Embeddings

Example: brain dynamics



Phase-Space Reconstruction Delay-Embeddings

Dynamical Invariants

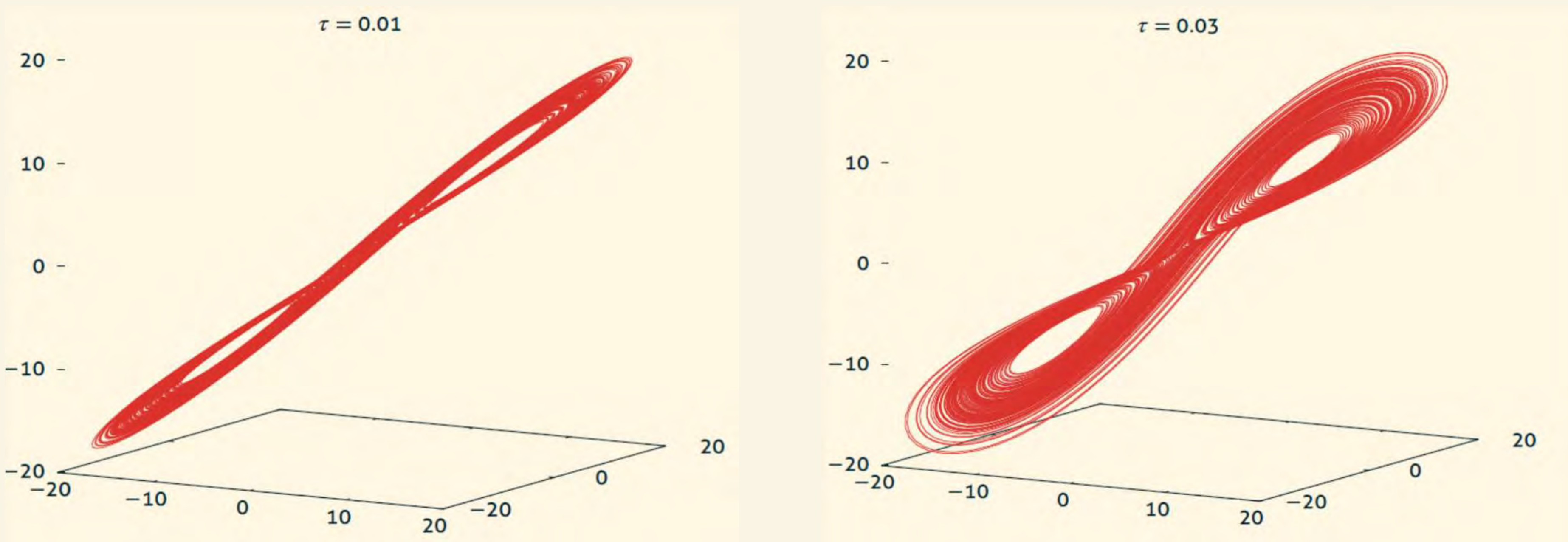
Important characteristics of the dynamics are invariant under the embedding transformation:

- Lyapunov exponents
- dimensions
- entropy
- …

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

example: Lorenz attractor



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Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

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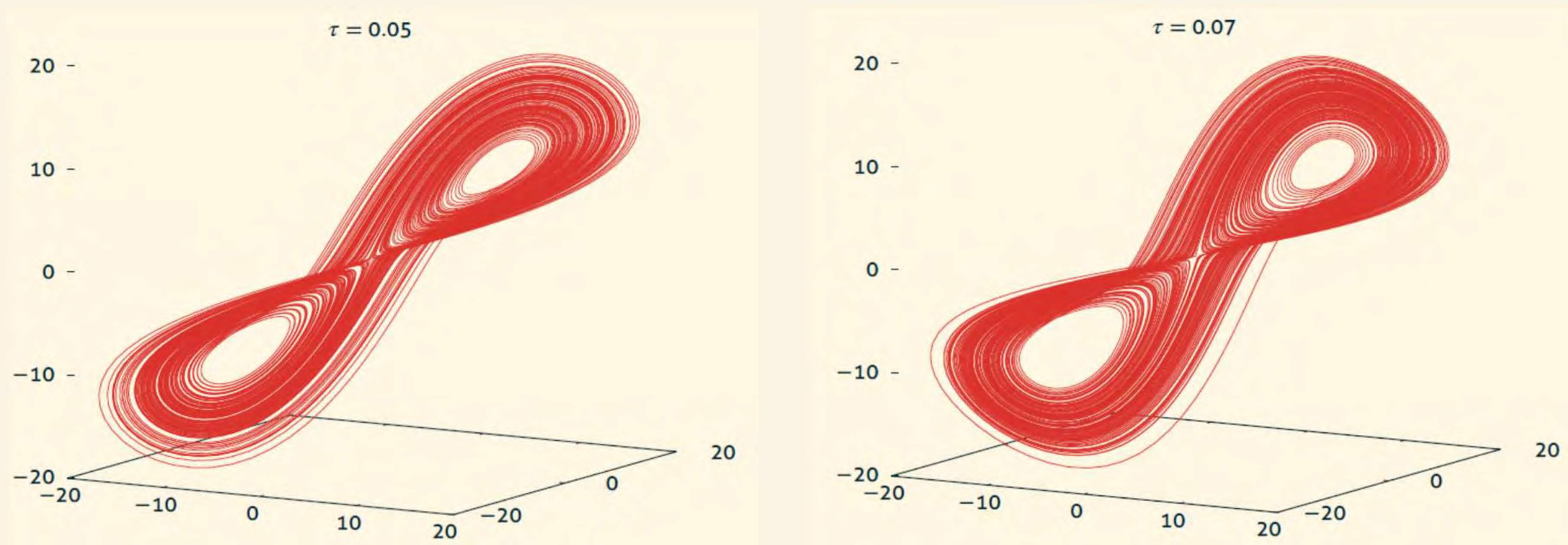

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Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

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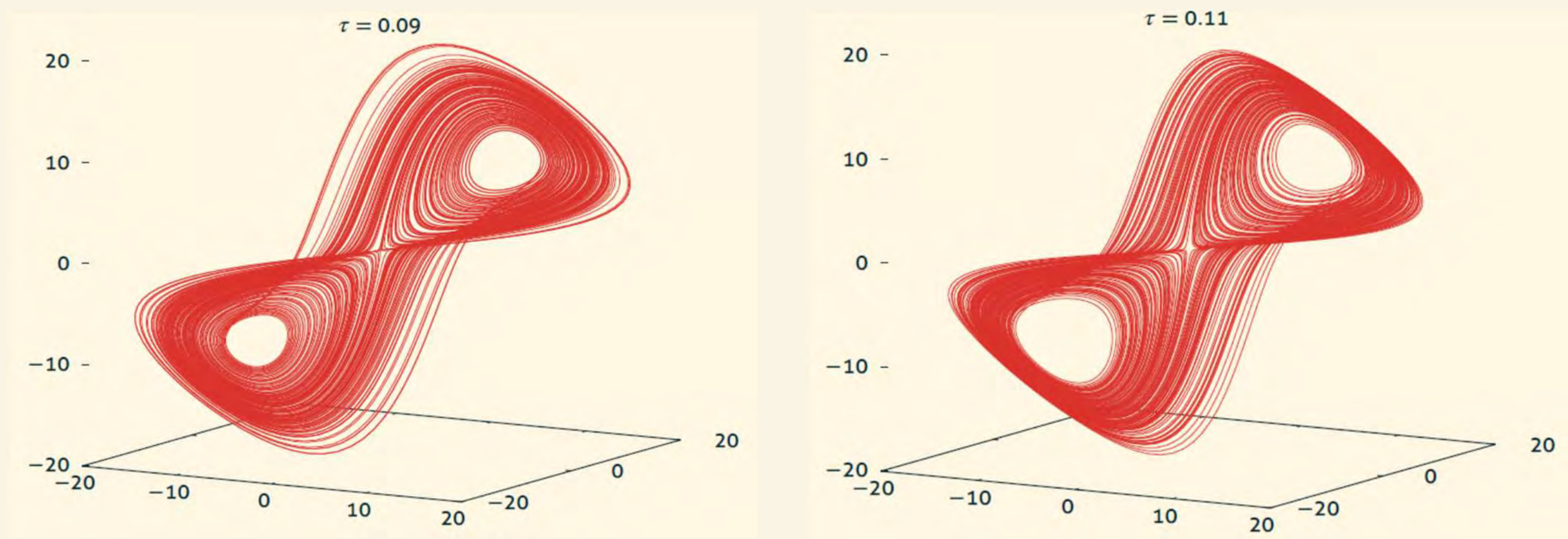

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Phase-Space Reconstruction Delay-Embeddings

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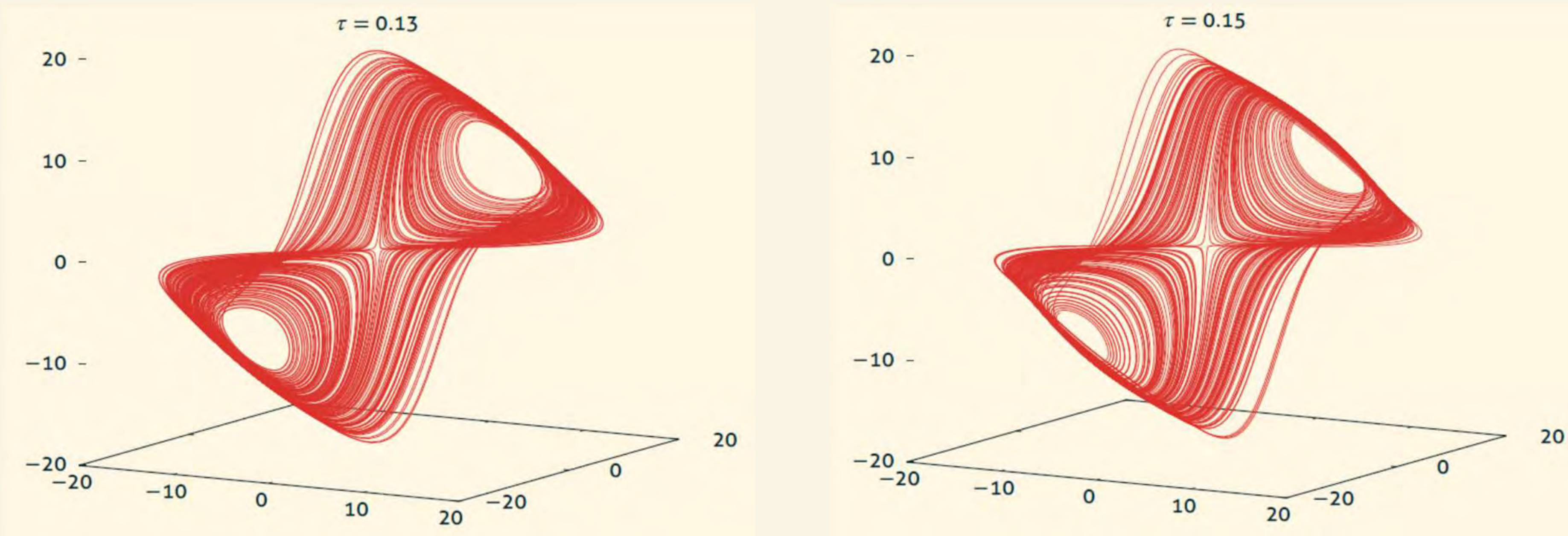

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Phase-Space Reconstruction Delay-Embeddings

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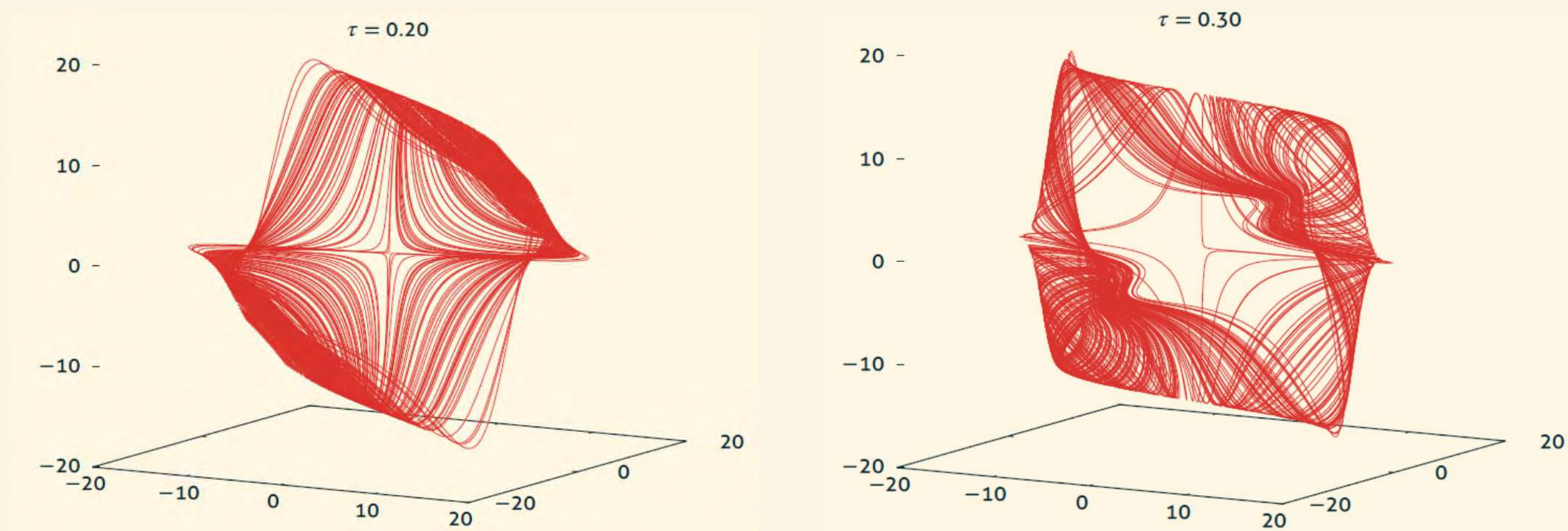

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Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

requirement for an embedding:
$(v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau})$ not fully redundant

→ aforementioned theorems: almost every τ yields an embedding:

requirements for a **good embedding**:

- minimum redundancy of $(v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau})$ (to unfold the attractor)
- reasonably small τ (to avoid folding the attractor onto itself)

(compare to: linear independence vs. orthogonality)

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

using zeros of the autocorrelation

Idea:

if autocorrelation = 0 for some delay $\Delta \Rightarrow$
v_t and $\mathrm{v}_{t+\Delta}$ are **linearly** independent on average

$\rightarrow$ choose the first zero of the autocorrelation Δ as embedding delay

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

using the first minimum of mutual information I

Idea: if common information for some delay Δ is minimum $\Rightarrow$

v_t and $\mathrm{v}_{t\text{-}\Delta}$ are independent on average (also includes **nonlinear** relationships)

$$I(M_1, M_2) = H(M_1) - H(M_1|M_2) = H(M_1) + H(M_2) - H(M_1, M_2)$$

where M_1 and M_2 denote measurements at times t and t-Δ, and $H = -\sum_i p_i \log p_i$ is the Shannon entropy

$\rightarrow$ choose the first minimum of the mutual information Δ as embedding delay

A. Fraser and H. Swinney, *Independent coordinates for strange attractors from mutual information,* Phys. Rev. A 33 (1986)

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

first minimum of mutual information I

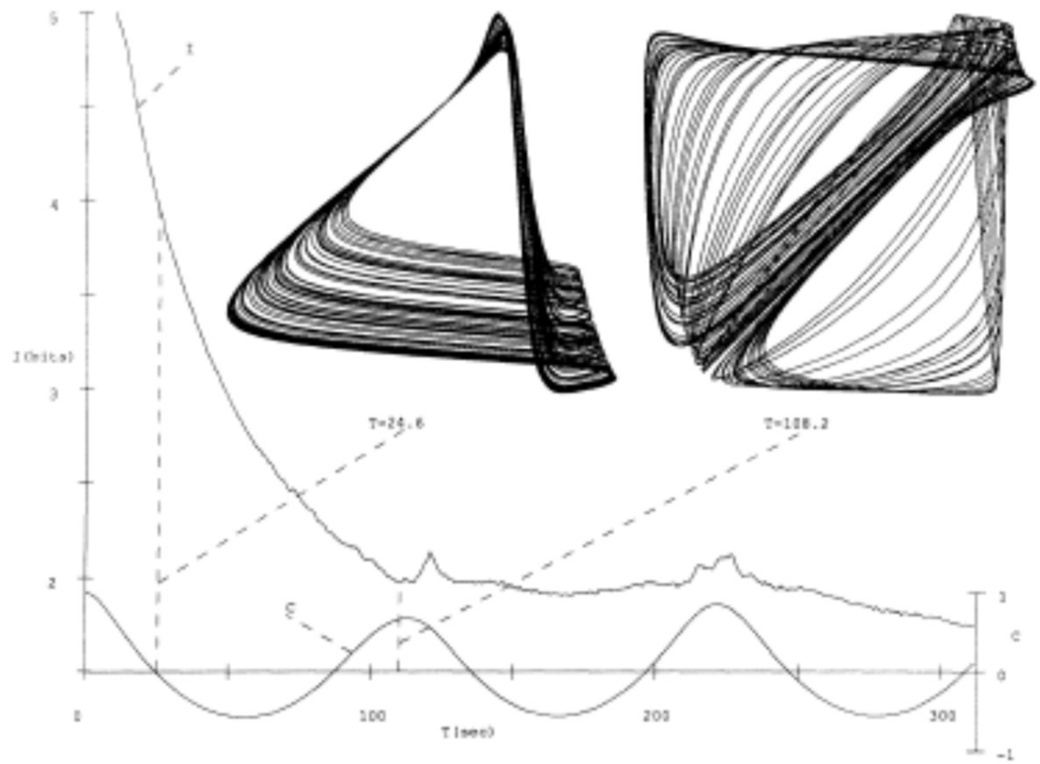

FIG. 1. Phase portraits of the Roux attractor (Ref. 3) in the Belousov-Zhabotinskii reaction. The dependence of the mutual information I and the autocorrelation function C on T are shown for calculations over 32 768 points. The coordinates used in constructing the portrait on the left are linearly independent (zero autocorrelation), while the coordinates used in the portrait on the right are more generally independent (local minimum of mutual information).

A. Fraser and H. Swinney, *Independent coordinates for strange attractors from mutual information,* Phys. Rev. A 33 (1986)

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter delay

- zeros of the autocorrelation function
- minima of the mutual information
- many more

No method is perfect or commonly agreed upon.

Practically:
- try at least two methods
- judge by further analysis
- alternative for $m \leq 3$: visually inspect the attractor
- keep embedding window $(m - 1)\tau$ (time span in an embedded vector) constant

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter dimension

embedding theorems define "sufficient" embedding dimension m

problem: dimension of system under study usually unknown

choosing m overly high may hamper further analyses
(impact of noise, finite number of data points, computational complexity)

→ Need other ways to determine a good m

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter dimension

Linear Dependence

idea:

- m is higher than necessary

⇒ attractor only covers a subspace of reconstruction space
(e.g., circle in $m = 3$)

- check whether embedded vectors have full rank.

difficulties:

- noise acts in all directions
- assumes linear dependence → dependence may be nonlinear

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter dimension

Asymptotic Invariants

idea:

- m too small ⇒ wrong dynamical invariants (in general)
- m sufficient ⇒ correct dynamical invariants

→ increase m until dynamical invariants converge

difficulties:

- criterion for convergence under real conditions (noise, finite number of data points, …)
- *wrong (in general)* is unpredictable

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter dimension

False Nearest Neighbors

idea:

• m too small ⇒ trajectories intersect

⇒ points close in reconstruction space that aren't close in actual phase space (false nearest neighbors)

→ increase m until false nearest neighbors vanish.

M. B. Kennel, R. Brown, and H. D. I. Abarbanel, *Determining minimum embedding dimension using a geometrical construction,* Phys. Rev. A 45 (1992)

Phase-Space Reconstruction Delay-Embeddings

Identifying embedding parameter dimension

False Nearest Neighbors

practically:

- choose threshold ε for nearest neighbors
- $NN(m)$: number of pairs of points in m-dimensional reconstruction space that are closer than ε
- $NN(m + 1) < NN(m)$

$\Rightarrow$ at least $NN(m) - NN(m + 1)$ false nearest neighbors in the m-dimensional reconstruction.

difficulties:

number of true nearest neighbors large and fluctuating (noise).

Phase-Space Reconstruction Delay-Embeddings

Summary

delay embedding allows to reconstruct attractor from single observable

- parameters m and τ have to be carefully chosen
- reconstructed phase space may be used for:
  - understanding
  - prediction
  - modelling
  - …

- characteristics preserved by reconstruction:
  dimensions, Lyapunov exponents, entropy, …

Phase-Space Reconstruction Delay-Embeddings

Extensions

- multivariate time series
- different embedding delays for each component
- state-dependent embedding delays