Lyapunov Exponents

Stability and Predictability

from Time Series

Long-term Behavior and Stability

long-term:

$t \rightarrow \infty$ (can not be achieved when observing real systems)

observation time:

$T << \infty$

largest characteristic time scale of system

$t_c < T$

Long-term Behavior and Stability

different types of long-term behavior:

- *unlimited growth*
  in practice: can usually not be observed
  in model studies: temporal stabilization or change of model

- *bounded dynamics*
  fixed point, equilibrium
  periodic, quasi-periodic motion
  chaotic motion

Q: how stable is the dynamics,
when perturbing the system?
(when changing control parameters? mostly not considered)

Long-term Behavior and Stability

stability dogma (Andronov & Pontryagin, 1930s):

“since all mathematical models are simplifications and abstractions, models that are relevant for applications must be structurally stable”

↓ however

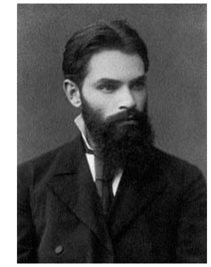
simple models that are composed of physically acceptable unit are structurally unstable

↓ (cf. weak/strong causality)

which (initial) states lead to the same / a similar long-term behavior?

→ concept of **Lyapunov-stability**

Long-term Behavior and Stability


A.M. Lyapunov

Consider an, in general, nonlinear dynamical system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}(t), \beta), \quad \boldsymbol{x} \in \mathbb{R}^d$$

Suppose f has an equilibrium at $\boldsymbol{x}_e$ so that $f(\boldsymbol{x}_e) = 0$, then this equilibrium

- is *Lyapunov stable*, if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$, such that if $\|\boldsymbol{x}(t=0) - \boldsymbol{x}_e\| < \delta$ then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < \epsilon$ for every $t \geq 0$,
- is *asymptotically stable*, if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|\boldsymbol{x}(t=0) - \boldsymbol{x}_e\| < \delta$, then $\lim_{t\to\infty} \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| = 0$,
- is *exponentially stable*, if it is asymptotically stable and there exist $\alpha > 0, \gamma > 0, \delta > 0$ such that if $\|\boldsymbol{x}(t=0) - \boldsymbol{x}_e\| < \delta$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \alpha \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| e^{-\gamma t}$, for all $t \geq 0$,

where $\|\cdot\|$ denotes, e.g., the Euclidean or the Manhattan norm.

A. M. Lyapunov, The general problem of motion stability, Ann. Math. Stud. 17 (1892).

Long-term Behavior and Stability

The aforementioned notions of *equilibrium stability* can be generalized to *orbital stability* (closed trajectory; i.e., periodic, quasi-periodic, or non-periodic orbit):

A trajectory $\Phi(t)$ is called *Lyapunov-stable* if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$, such that the trajectory of any solution $\boldsymbol{x}(t)$ starting at the δ-neighborhood of $\Phi(t)$ remains in the ϵ-neighborhood of $\Phi(t)$ for all $t \geq 0$.

Linear stability analysis:

- consider small perturbation (of equilibrium/trajectory)
- expand f in Taylor-series
- check eigenvalues of Jacobian; stable, if all have strictly negative parts
- real part of the largest eigenvalue (**Lyapunov exponent**) determines time to return to equilibrium/trajectory after perturbation

Long-term Behavior and Stability

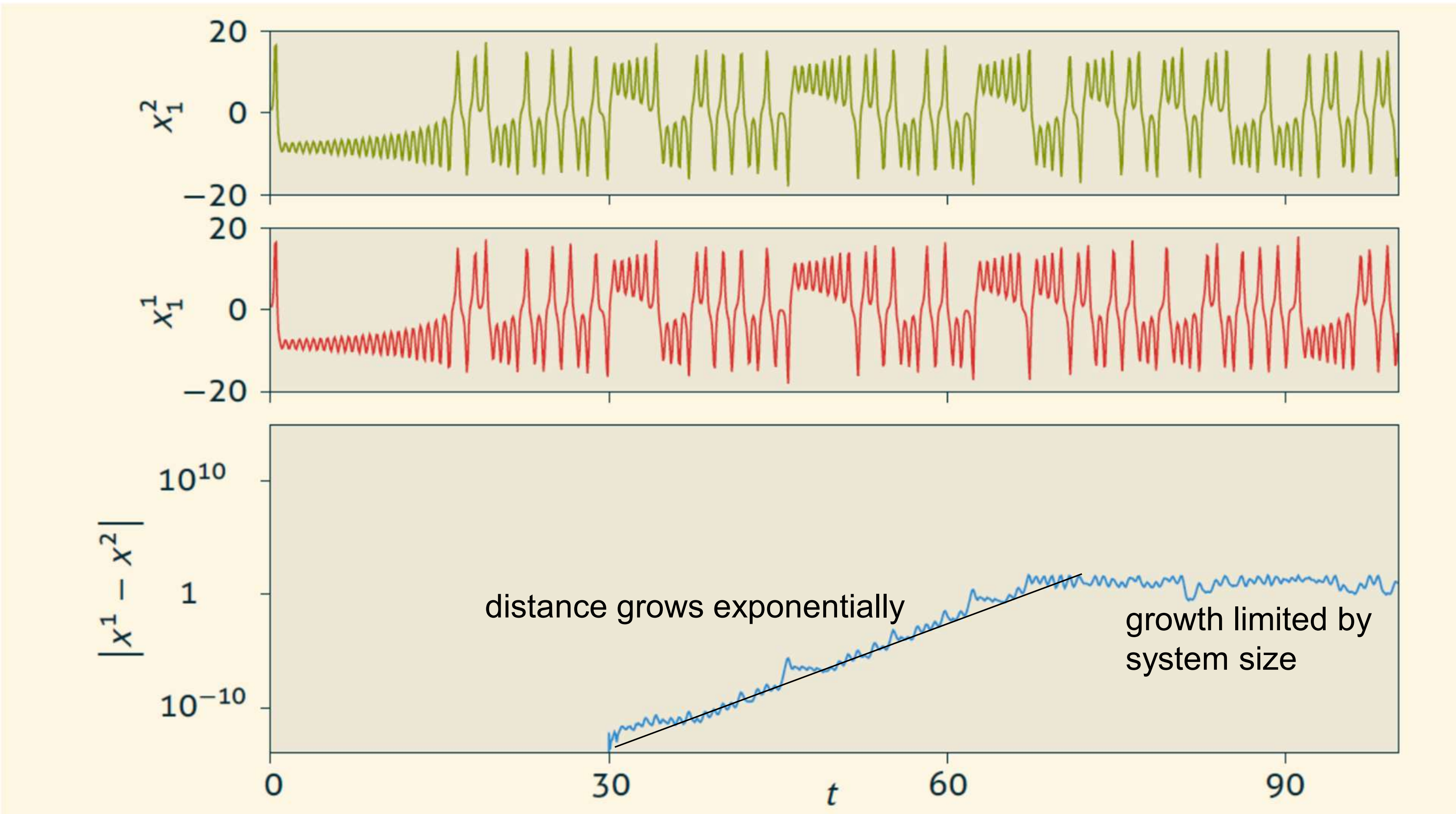
chaotic motion is (locally) Lyapunov-unstable:

divergence:
distance between initially close trajectory segments grows exponentially in time (stretching mechanism)

convergence:
divergence of initially close segments limited by system size; when reached, distance shrinks again (folding mechanism)

Long-term Behavior and Stability

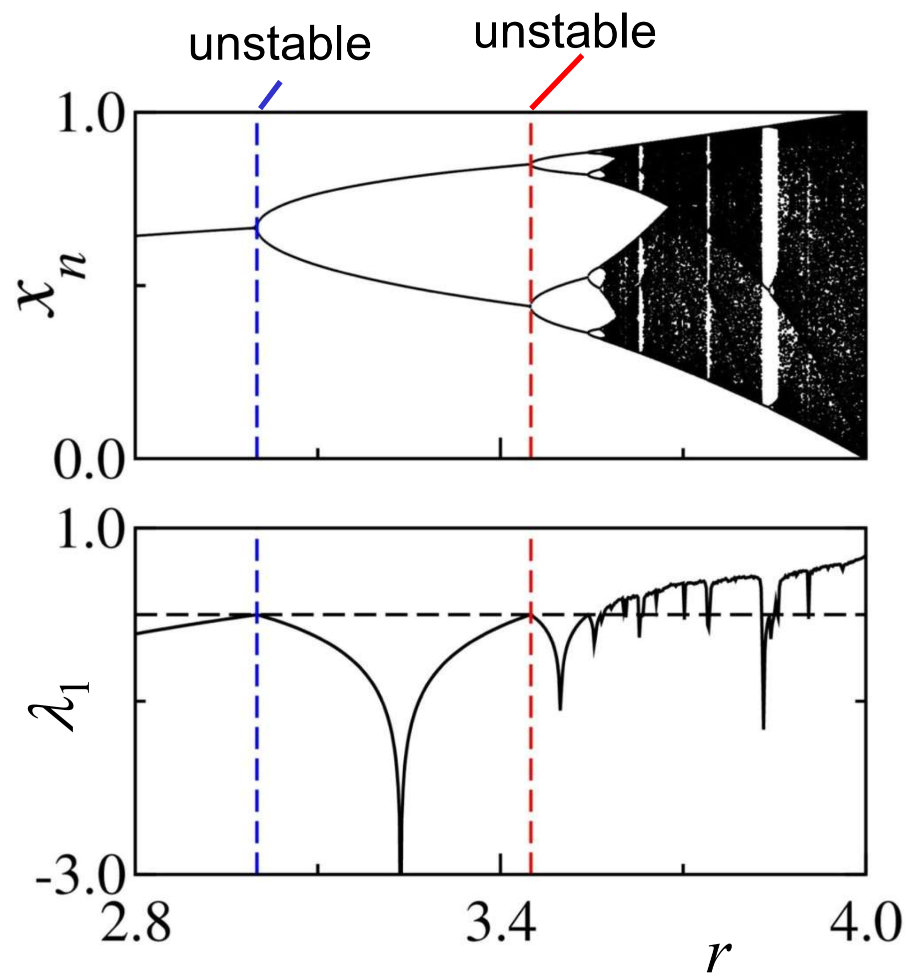
Example: two identical Lorenz oscillators with initial conditions; one oscillator is slightly perturbed (10^{-14}) at *t* = 30



from: G. Ansmann

Long-term Behavior and Stability

Example: logistic map: $x_{n+1} = rx_n(1 - x_n); \quad x_n \in [0,1]; \quad r \in [0,4]$



λ_1 = largest Lyapunov exponent (dominates the dynamics)

Long-term Behavior and Stability

spectrum of Lyapunov exponents: λ_i with $i = 1, \ldots, d$

characterize growth rates in different local directions of phase space

Lyapunov exponents and divergence: $\sum_{i=1}^{d} \lambda_i = \nabla \cdot f$

dissipative system: $\sum_{i=1}^{d} \lambda_i < 0$

largest Lyapunov exponent:

$$\lambda_1 = 0 \rightarrow \text{regular dynamics}$$
$$\lambda_1 > 0 \rightarrow \text{chaotic dynamics}$$
$$\lambda_1 < 0 \rightarrow \text{fixed-point dynamics}$$
$$\lambda_1 \rightarrow \infty \rightarrow \text{stochastic dynamics}$$

Lyapunov exponents from time series

model:	***field data:***
continuous trajectories	→ discrete trajectories
actual phase space	→ reconstruction
evolution of arbitrary states	→ available trajectories
equations of motion	→ available trajectories

and of course: finite data, noise, …

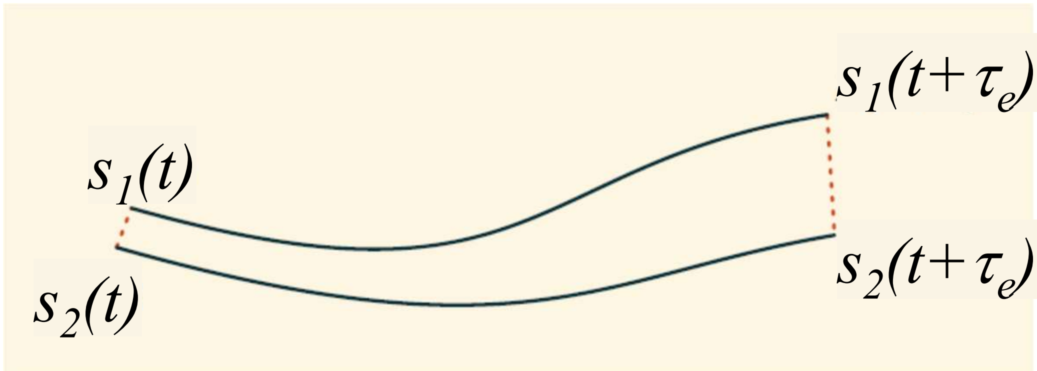
concepts and algorithms (most widely used)***:***

- spectrum of Lyapunov exponents (in general, hard to estimate)
  (Sano & Sawada, 1985; Eckmann et al., 1986; Stoop & Parisi, 1991)

- largest Lyapunov exponent
  (Wolf et al. 1985; Rosenstein et al., 1993; Kantz, 1994)

Largest Lyapunov exponent **Idea**

Consider evolution of two nearby trajectory segments s_1 and s_2



For infinitesimally close trajectory segments $(||s_1(t) - s_2(t)|| \to 0)$
and for infinite time evolution ($\tau_e \to \infty$)
the distance between segments grows or shrinks exponentially:

$$||s_1(t+\tau_e) - s_2(t+\tau_e)|| = ||s_1(t) - s_2(t)||\, \mathrm{e}^{\lambda_1 \tau_e}$$

Largest Lyapunov exponent **Idea**

$$||s_1(t+\tau_e) - s_2(t+\tau_e)|| = ||s_1(t) - s_2(t)||\, \mathrm{e}^{\lambda_1 \tau_e}$$

Solve for λ_1 and implement the limits.

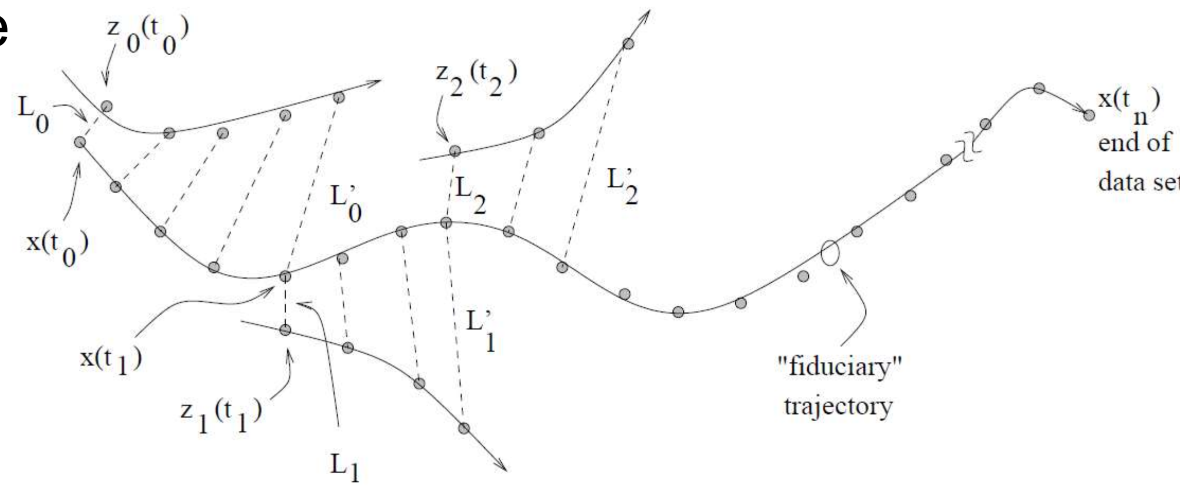
Let s_1 and s_2 denote two near trajectory segments of the dynamics. The first Lyapunov exponent is defined as:

$$\lambda_1 := \lim_{\tau_e \to \infty} \lim_{||s_1(t) - s_2(t)|| \to 0} \frac{1}{\tau_e} \ln \left(\frac{||s_1(t+\tau_e) - s_2(t+\tau_e)||}{||s_1(t) - s_2(t)||} \right)$$

Also: *largest Lyapunov exponent* or just *Lyapunov exponent*.

Largest Lyapunov exponent Wolf-Algorithm

estimates the dominant Lyapunov exponent from a time series by monitoring orbital divergence



1. reconstruct phase space
2. pick $x(t_0)$ on *fiduciary* trajectory
3. find nearest neighbor $z_0(t_0)$
4. compute $||z_0(t_0) - x(t_0)|| =: L_0$
5. follow *difference trajectory* (dashed line forwards in time and compute $||z_0(t_i) - x(t_i)|| =: L_0(i)$. Increment i until $L_0(i) > \epsilon$, call that value L'_0 and that time t_1
6. find $z_1(t_1)$, the *nearest neighbor* of $x(t_1)$, and loop to step 4. Repeat procedure to the end of *fiduciary* trajectory ($t = t_n$). Keep track of the L_i and L'_i

Find largest (positive) Lyapunov exponent from:

$$\lambda_1 \approx \frac{1}{N\Delta t} \sum_{i}^{M-1} \log_2 \left(\frac{L'_i}{L_i} \right)$$

where M denotes number of loops, and N number of time steps on fiduciary trajectory; $N\Delta t = t_n - t_0$

Wolf, A., Swift, J. B., Swinney, H. L., & Vastano, J. A. (1985). Determining Lyapunov exponents from a time series. Physica D 16(3), 285-317.

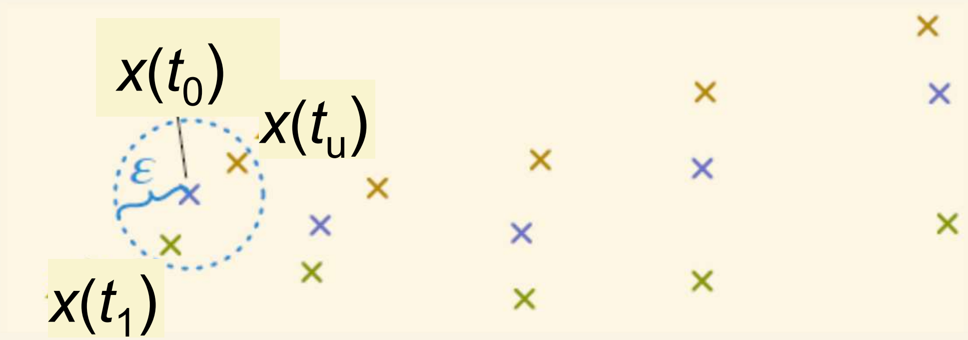
Largest Lyapunov exponent — Wolf-Algorithm

Limitations

- too many parameters that have to be chosen a priori
- problems may be obfuscated:
  no exponential growth due to noise
  embedding dimension *m* too small
- highly sensitive to noise
- difficult to find neighboring trajectory segment with required properties

→ need a different way to ensure alignment
to direction of largest growth

Largest Lyapunov exponent Rosenstein-Kantz Algorithm



from: G. Ansmann

1. choose reference state $x(t_0)$ and all states $x(t_1)$, …, $x(t_u)$ in ε-neighborhood

2. for given τ_e, define average distance of respective trajectory segments from the initial one as

$$s(t,\tau_e) := \frac{1}{u}\sum_{j=1}^{u} ||x(t_0+\tau_e) - x(t_j+\tau_e)||$$

3. average over all states as reference states:

$$S(\tau_e) := \frac{1}{N}\sum_{t=1}^{N} s(t,\tau_e)$$

4. obtain largest Lyapunov exponent from region of exponential growth of $S(\tau_e)$

Largest Lyapunov exponent Rosenstein-Kantz Algorithm

82 H. Kantz / Physics Letters A 185 (1994) 77–87

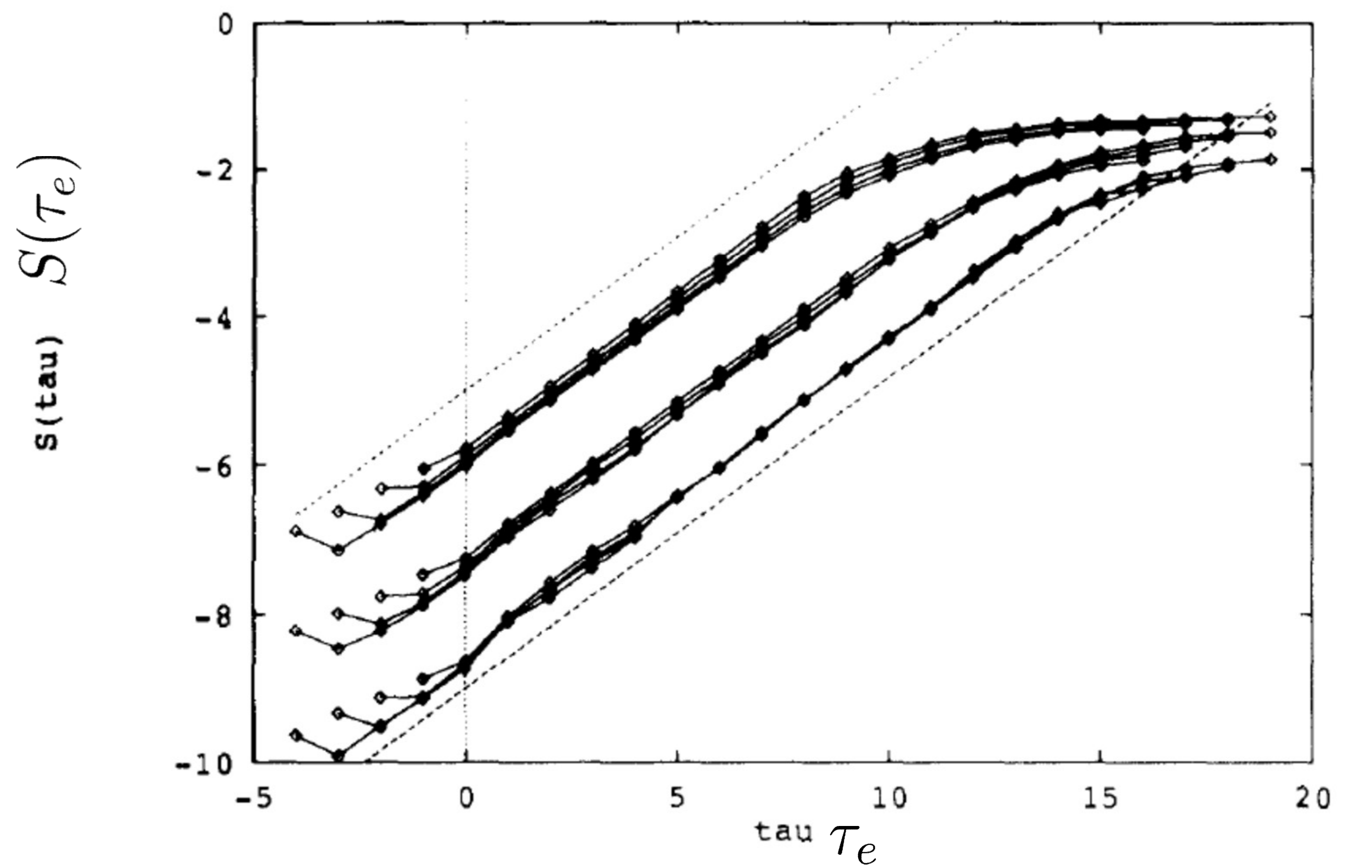


Fig. 2. $S(\tau)$ for a Hénon trajectory of length 2000. The different curves correspond to ϵ = 0.0005, 0.002 and 0.008 (the three bunches from bottom to top) and embedding dimension m = 2–5. The dashed lines have slopes λ_{exact} = 0.4169. Again $\tau \leqslant 0$ corresponds to the components used to define the local neighbourhoods.

Largest Lyapunov exponent Rosenstein-Kantz Algorithm

Mind how you average

1. average over the neighborhood of a reference state → $s(t, \tau_e)$
2. average $s(t, \tau_e)$ over all reference states → $S(\tau_e)$
3. Obtain λ_1 from slope of $S(\tau_e)$

Density of states in a region of the attractor affects:

- reference states
- states in neighborhood of given reference state

Separating averaging in steps 1 and 2
(instead of averaging of all ε-close pairs)
ensures that density is accounted for only once
(and not twice)

Largest Lyapunov exponent Rosenstein-Kantz Algorithm

Advantages and Problems

- region of exponential growth can be determined a posteriori
  (be careful of wishful thinking though)
- absence of exponential growth usually detectable
  (but only usually)
- region of strong noise influence can be detected and excluded
- can only determine the largest Lyapunov exponent

Largest Lyapunov exponent extensions

- tangent-space methods

→ require estimate of Jacobian

- spectrum of Lyapunov exponents

→ requires a lot of data

Largest Lyapunov exponent units

for flows: **inverse seconds**

for maps: **inverse iterations**

other choices: bits/second or bits/iteration

Spectrum of Lyapunov exponents and type of dynamics

For bounded, continuous-time dynamical systems, we have:

signs of Lyapunov exponents	dynamics
−, −−, −−−, …	fixed point
+, ++, +++, …, +0, ++0, …	not possible (unbounded)
0, 00, 000, …	no dynamics ($f = 0$)
0−, 0−−, 0−−−, …	periodic / limit cycle
00−, 00−−, 00−−−, …	quasiperiodic (torus)
000−, 0000−, …, 000−−, …	quasiperiodic (hypertorus)
+0−, +0−−, +0−−−, …	chaos
++0−, +++0−, …, ++0−−, …	hyperchaos
∞, …	noise

Largest Lyapunov exponent what can go wrong?

field applications

- number of data points ($\lim N \rightarrow \infty$)
- data precision
  adopt to requirement of small ε-neighborhood
- strong correlations in data (sampling interval)
  use Theiler correction (see Dimensions)
- noise
  similar impact as with Dimensions
- filtering
  classical filter affect negative Lyapunov exponents only
  due to adding a (passive) system → extra Lyapunov exponent
  magnitude ~ cutoff frequency

Largest Lyapunov exponent what can go wrong?

False indications of chaos:

- unbounded orbits can have $\lambda_1 > 0$

- orbits can separate but not exponentially

(check boundedness and be sure orbit has adequately sampled attractor; check for contraction to zero within machine precision)

- can have **transient chaos***

(double-check with other methods)

*e.g., Y.-C. Lai and T. Tél, Transient Chaos, Complex Dynamics on Finite-Time Scales (Springer, New York, 2011)

Largest Lyapunov exponent transient chaos: an example

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Self-Induced Switchings between Multiple Space-Time Patterns on Complex Networks of Excitable Units

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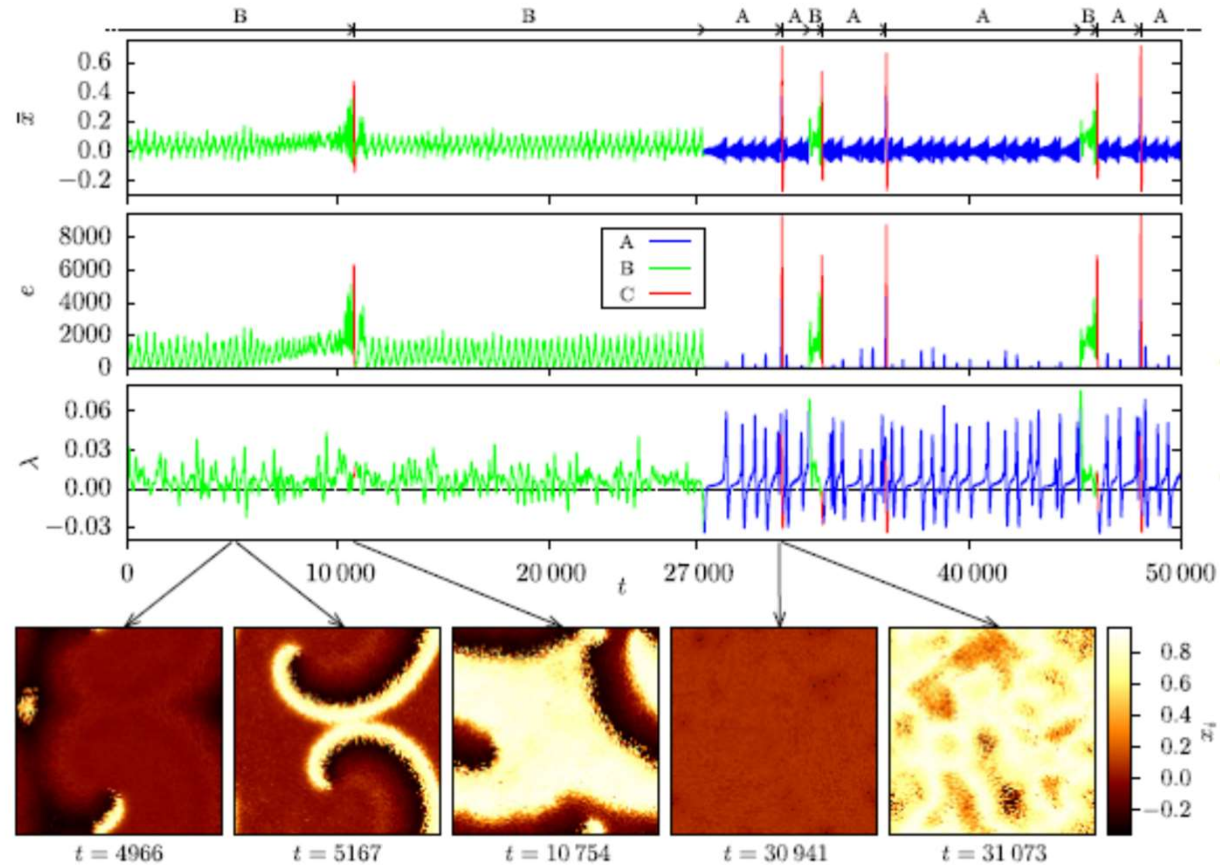
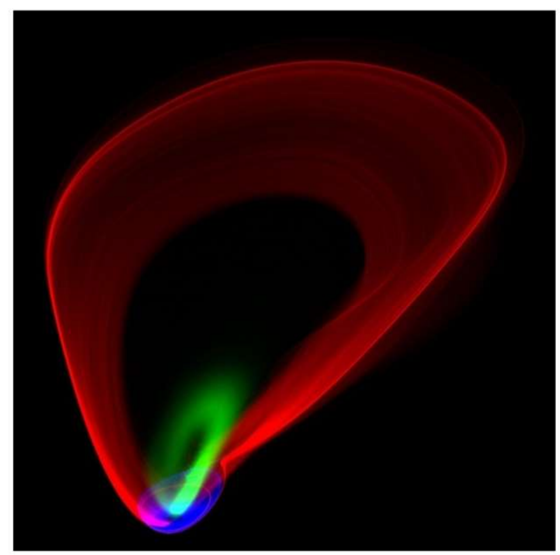


FIG. 1. First to third row: Exemplary temporal evolutions of $\bar{x}$, of the number e of units with $x_i > 0.4$ ("excited units"), and of an estimate λ of the largest local Lyapunov exponent (temporal evolution smoothed with a Gaussian kernel with a width of 30 to improve readability). The line colors indicate the patterns as automatically classified (blue: A, low-amplitude oscillations; green: B, waves; red: C, extreme events). These patterns are also indicated at the very top with pattern C being indicated with a vertical line. Bottom: Snapshots of the spatial distribution of $x_i(t)$ at times corresponding to selected local minima and maxima of $\bar{x}$ (from left to right): adjacent minimum and maximum around $t = 5000$, the maximum during the event around $t = 10000$, the minimum before the event around $t = 30\,000$, and the maximum during that event. Units are represented by pixels, which are arranged according to the lattice underlying the small-world network and whose color encodes the value of the respective x_i [65].

$$\dot{x}_i = x_i(a - x_i)(x_i - 1) - y_i + \frac{k}{m}\sum_{j=1}^{n} A_{ij}(x_j - x_i),$$

$$\dot{y}_i = b_i x_i - c y_i.$$



Largest Lyapunov exponent Interpretation

- stability and type of the dynamics:

  $\lambda_1 > 0$ chaos, unstable dynamics

  $\lambda_1 = 0$ regular dynamics

  $\lambda_1 < 0$ fixed-point dynamics

- quantification of loss of information due to action of nonlinearity

- prediction horizon:

$$T_{\mathrm{p}} \approx \frac{-\ln(\rho)}{\sum_{i,\lambda_i>0} \lambda_i}$$

where:

ρ denotes accuracy of measurement (initial state)

$\sum_{i,\lambda_i>0}$ is sum of positive Lyapunov exponents

Kaplan-Yorke conjecture

relationship between dimension and Lyapunov exponents

$$D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}, \quad \text{where} \ \sum_{i=1}^{k} \lambda_i \geq 0 \ \text{ and } \ \sum_{i=1}^{k+1} \lambda_i < 0$$

Kaplan-Yorke dimension D_{KY} equals information dimension D_1
(Note: conjecture not generally valid!)

example:
- Hénon map with parameters a = 1.4 and b = 0.3
- λ_1 = 0.603, λ_2 = -2.34
- we find with k = 1:

$$D_{KY} = k + \frac{\lambda_1}{|\lambda_2|} = 1 + \frac{0.603}{|-2.34|} = 1.26$$

Kaplan-Yorke conjecture

relationship between dimension and Lyapunov exponents