Testing for Determinism

in Time Series

Brief Recap: Dynamical Invariants and System Properties

invariants	***properties***
dimensions	scaling behavior, self-similarity, number of degrees of freedom, complexity, nonlinearity (from fractality), determinism
Lyapunov exponents	stability (short- and long-term), predictability, determinism, nonlinearity, chaos
entropies	(dis-)order, complexity, predictability, determinism, nonlinearity, chaos

Brief Recap: Dynamical Invariants and Type of Dynamics

dynamics	***properties***	***invariants***
regular	deterministic long-term predictability strong causality	$D \in \mathbb{N}$ $\lambda_1 = K = 0$
chaotic	deterministic limited predictability violation of strong causality nonlinearity	$D \notin \mathbb{N}$ $0 < (\lambda_1, K) \ll \infty$
stochastic	randomness no predictability	$(D, \lambda_1, K) \to \infty$

Brief Recap: Dynamical Invariants and Real-World Data

When analyzing time series from real-world systems

- many prerequisites can not strictly be fulfilled
- limited significance of dynamical invariants
- cannot strictly proof chaos, nonlinearity, deterministic structure

→ need other methods to
- test for determinism
- test for nonlinearity

determinism / stochasticity

Wold decomposition

"every weak stationary time series *v(t)* can be written as the sum of two time series:
a *linearly deterministic* *d(t)* and a *stochastic* *ε(t)*"

$$v(t) = d(t) + \sum_{j=0}^{\infty} b_j \epsilon(t-j)$$

$$\sum_{j=0}^{\infty} |b_j|^2 < \infty$$

recap weak stationarity (see Linear Methods):
mean, variance, and covariance do not depend on time:

$$\mathrm{E}\left(v(t)\right) = \mu;\ \mathrm{Var}\left(v(t)\right) = \sigma^2;\ \mathrm{Cov}\left(v(t), v(t+\Delta)\right) = \gamma(\Delta)$$

$$v(t) = d(t) + \sum_{j=0}^{\infty} b_j \epsilon(t-j)$$

H. O. Wold. *The analysis of stationary time series*. Almqvist & Wiksell, Uppsala, 1954

Brief Intermezzo — testing for weak stationarity

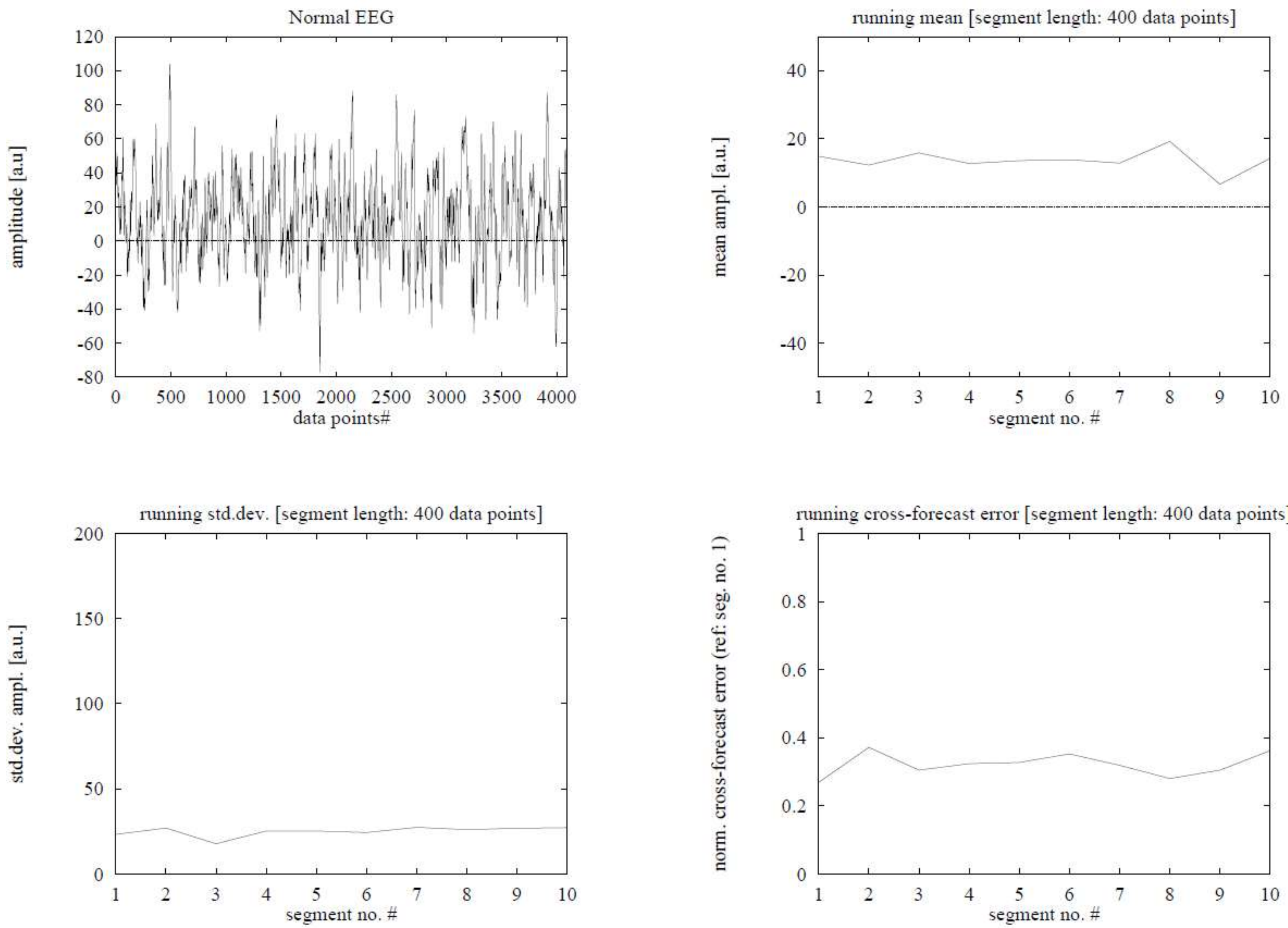
given time series $\{v_n; n = 1, \ldots, N\}$
split it into contiguous segments of given length $S_i^{(l)} = \left\{v_{(i-1)l+1}, \ldots, v_{il}\right\}$

estimate for each segment
- scalar statistics: mean, variance (std. dev.)
- more elaborate statistics: cross-forecast error (phase-space-based one-step-ahead predictor*)

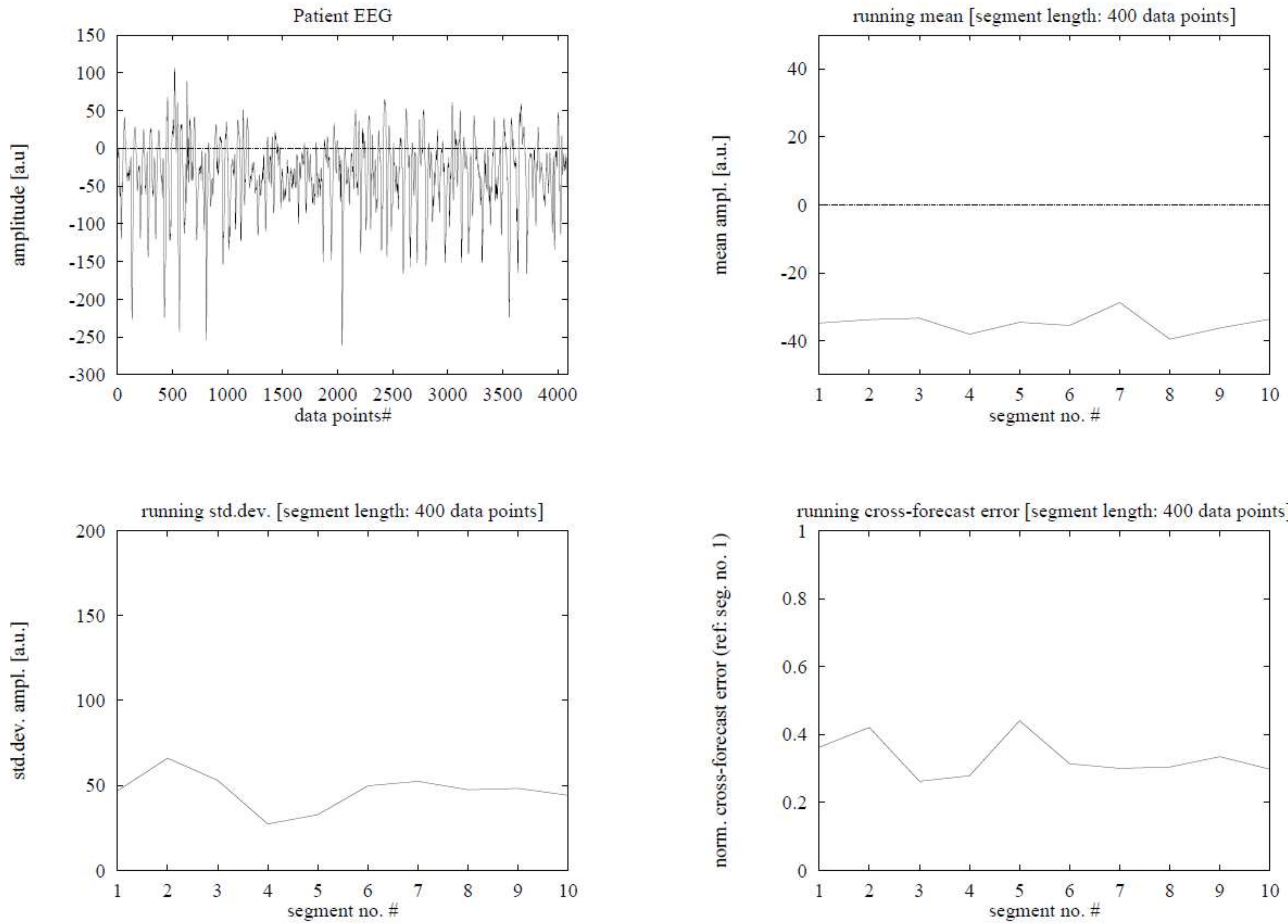
problem: what is an *appropriate* segment length?
- need to find a tradeoff between good statistics (large N) and hints for weak stationarity

*T. Schreiber, *Detecting and analysing nonstationarity in a time series using nonlinear cross predictions*, Phys. Rev. Lett. **78**, 843´, 1997

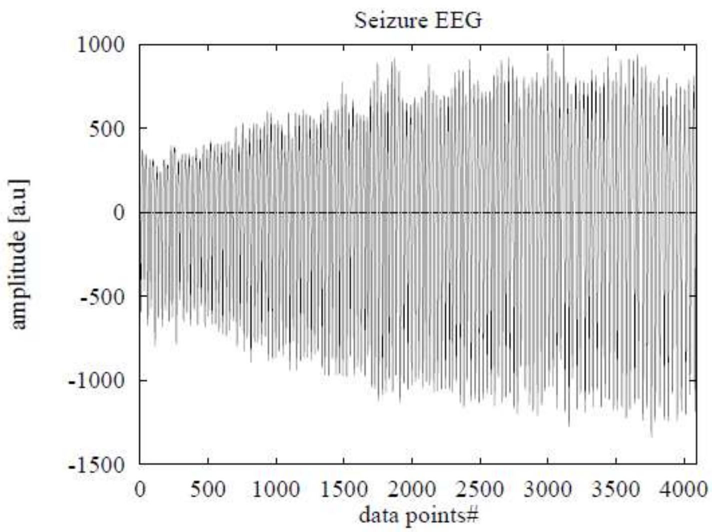
Brief Intermezzo testing for weak stationarity

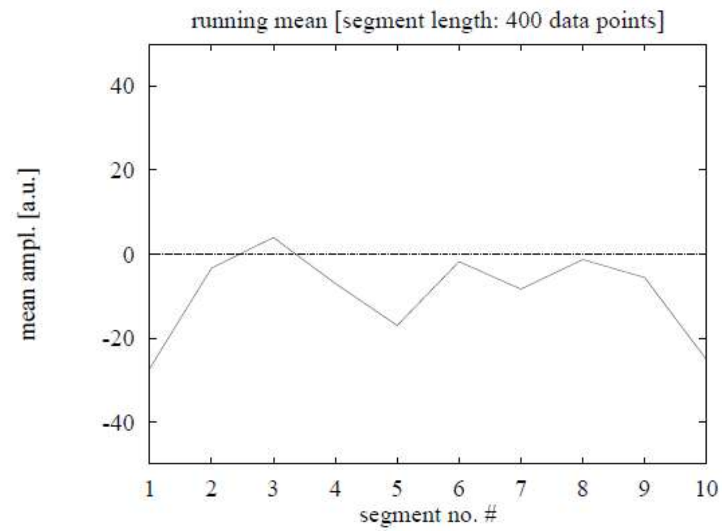


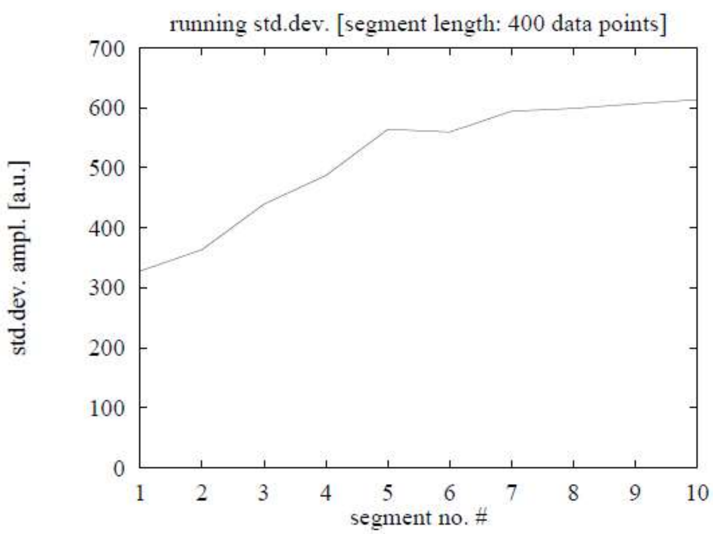
Brief Intermezzo testing for weak stationarity

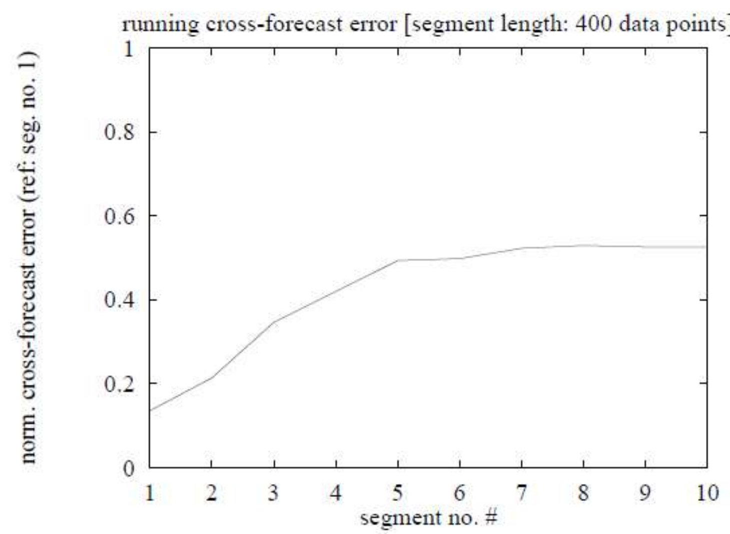


Brief Intermezzo testing for weak stationarity







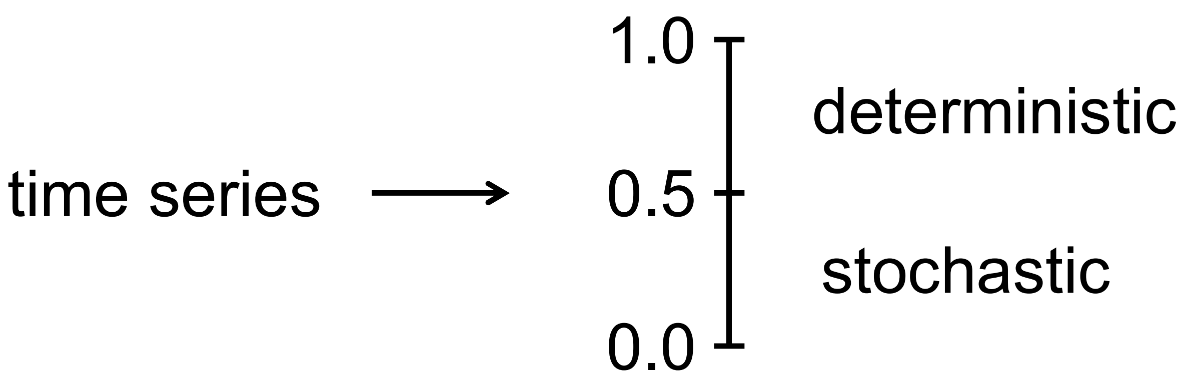


determinism / stochasticity — Wold decomposition

decomposition theorem

- assumes *linearly* deterministic component
- assumes additivity
- allows for “binary” decision only (either deterministic or stochastic)

desirable: mapping onto some interval



determinism / stochasticity idea

with *weak* causality criterion (*equal causes* → *equal effects*), we have:

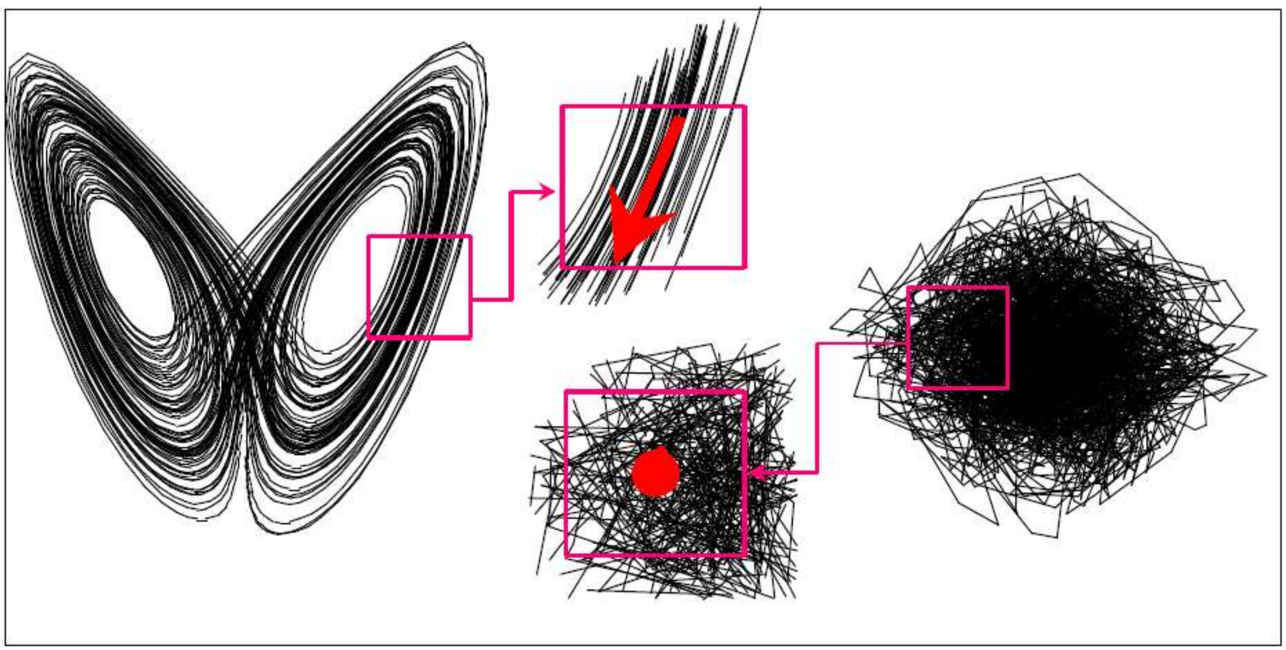
- motion in phase space is uniquely determined
- no self-crossing of trajectory
- with **smooth** equations of motion, we find that close trajectory segments are parallel
- in infinitesimal small volume elements, we find

$$\frac{\mathrm{d}\mathbf{x}_i}{\mathrm{d}t} \rightarrow \frac{\mathrm{d}\mathbf{x}_j}{\mathrm{d}t} \quad \forall \mathbf{x}_i \rightarrow \mathbf{x}_j$$

determinism / stochasticity idea

with *strong* causality criterion (*similar causes* → *similar effects*),
we have:

- trajectory segments are **aligned** in small (but finite) volume elements
- this defines a ***local flow in phase space****



* for strange attractors with "empty" regions this holds "on average" (over the whole attractor)

determinism from time series approaches

phase-space-based approaches to test for determinism in time series

vector field in phase-space

↓

parallelism*
(Kaplan & Glass, Phys Rev Lett 68, 427, 1992)

continuity
(Wayland et al, Phys Rev Lett 70, 580, 1993)

smoothness
(Salvino & Cawley, Phys Rev Lett 73, 1091, 1994)

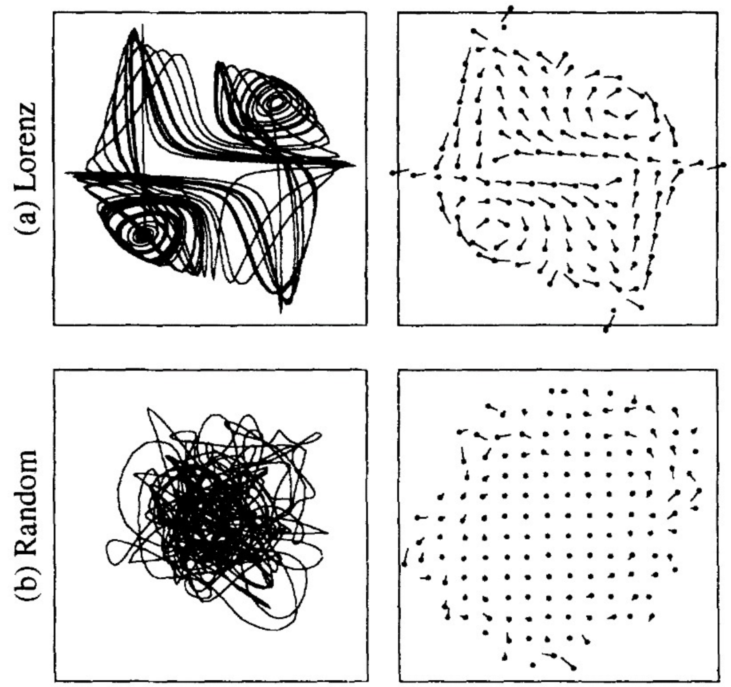
↓ ← statistics

determinism / stochasticity

* discussed here

determinism from time series

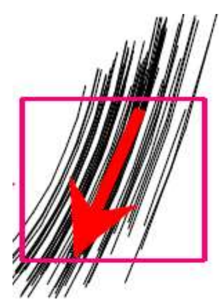
Kaplan-Glass approach



from: Kaplan & Glass, Phys Rev Lett 68, 427, 1992

observation:
the tangent to the trajectory generated by a deterministic system is a function of position in phase-space

determinism:
all tangents to the trajectory in a given region of phase-space will have similar orientations



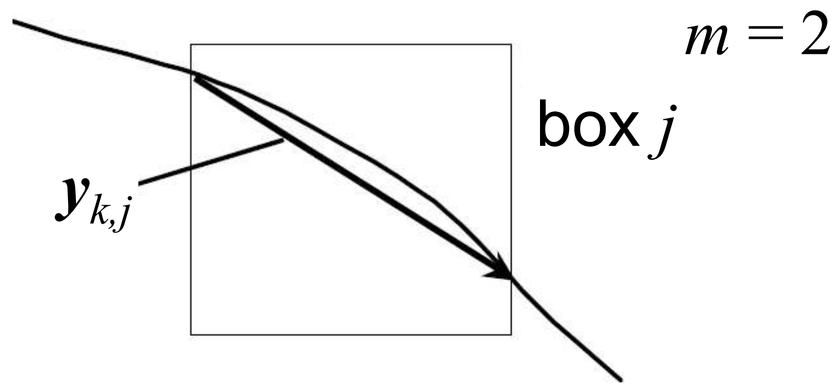
determinism from time series — Kaplan-Glass approach

- *phase-space reconstruction*
  (delay-embedding; embedding parameter chosen appropriately)

$$\left(v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau}\right)$$

- *coarse-graining* of phase-space (boxes with finite side length)

- k^{th} pass of trajectory through box j
  generates *trajectory vector* $\boldsymbol{y}_{k,j}$
- vector has unit length (normalized)
- vector orientation determined by
  vector between entry and exit points
  (mean direction; advantage:
  acts like low-pass filter)

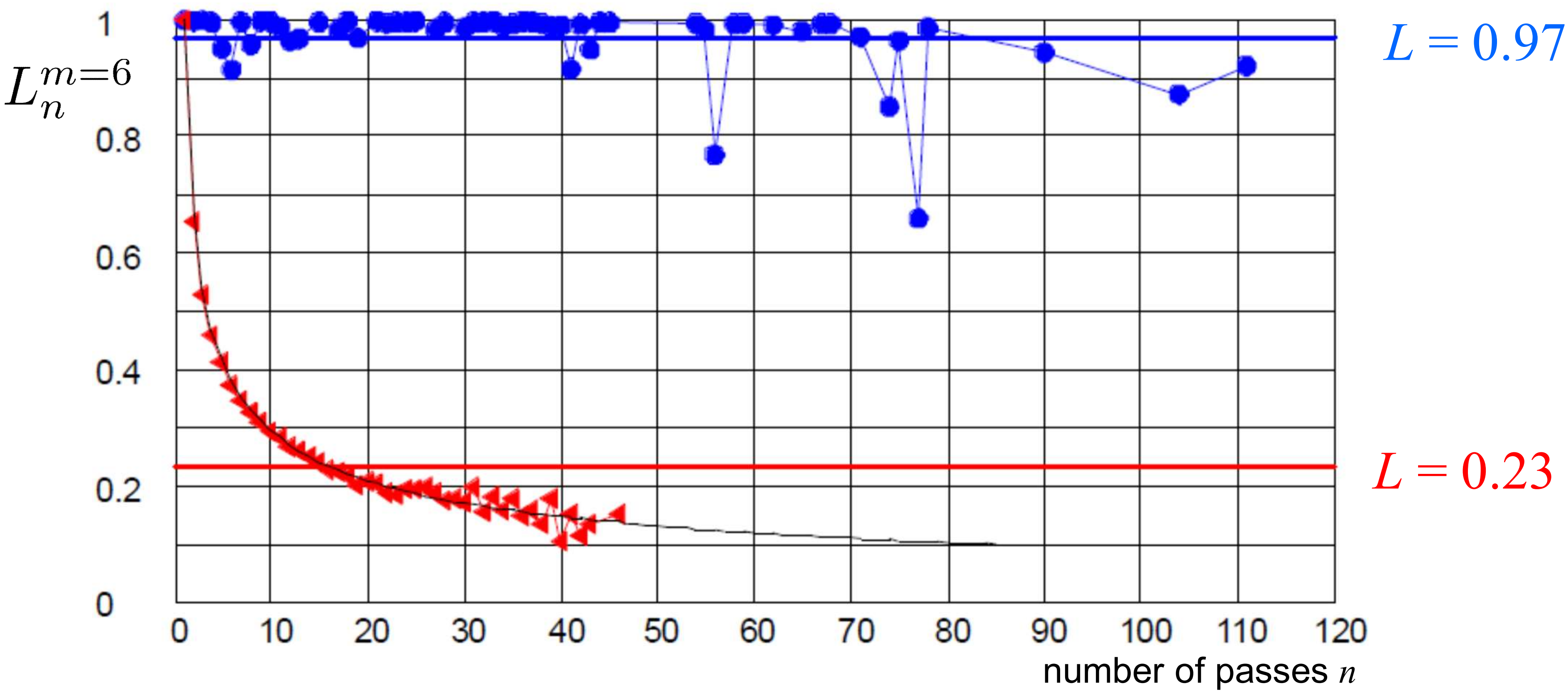


determinism from time series Kaplan-Glass approach

- *statistics over all trajectory vectors* and boxes
  n_j passes of trajectory through box j
  normalize all n_j trajectory vectors (passes are treated equally)
  vectorial summation and normalization by number of passes (boxes are treated equally)
  define mean trajectory vector as: $\mathbf{Y}^j_{n_j} := \frac{\sum_{k=1}^{n_j} \mathbf{y}_{k,j}}{n_j}$

- *characterize local flow in phase-space*
  consider global average over all boxes
  sort boxes according to the respective number of passes
  evaluate distribution of mean trajectory vector lengths dependent on number of passes
  define: $L_n^m := \left\langle \left| \mathbf{Y}^j_{n_j} \right| \right\rangle_{n=n_j}$

determinism from time series — Kaplan-Glass approach



Lorenz system

white noise

$$L := \left\langle L_n^{m=6} \right\rangle_{n>1}$$

determinism from time series — Kaplan-Glass approach

\- *reasons for deviations from expected values*

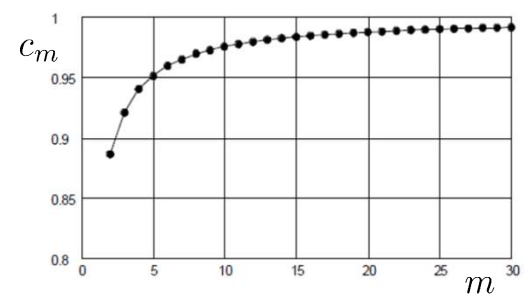
upper bound: in general, we have $\lim_{\epsilon \to 0} L_n^m = 1$
can't take limit, need coarse graining
if ε too small → too few passes → insufficient statistics

lower bound: consider graph for white noise
one can find: $L_n^m \propto \frac{1}{\sqrt{n}}$

due to n-step random walk in m dimensions*

$$R_n^m = \sqrt{\frac{2}{nm}} \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})} = c_m \frac{1}{\sqrt{n}}$$

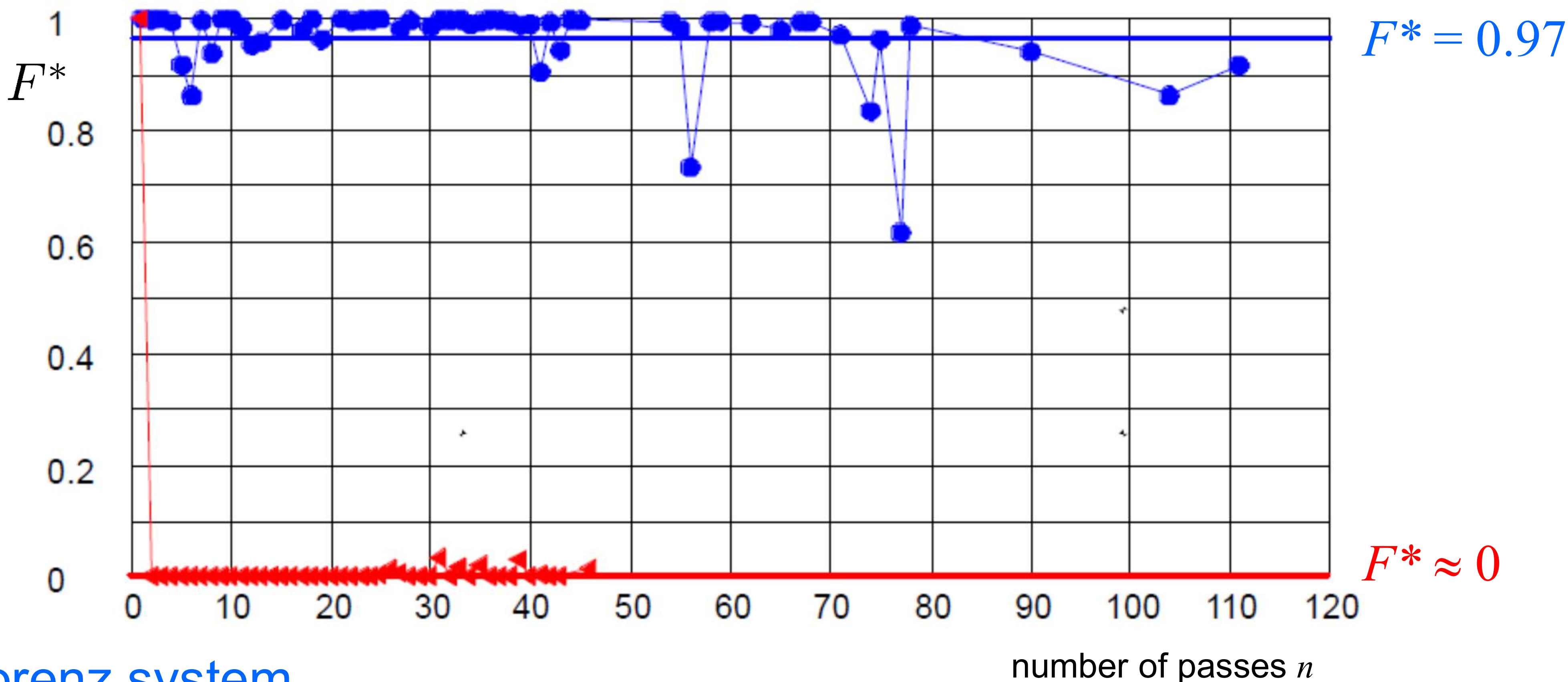
$$\lim_{m \to \infty} c_m = 1$$



RAYLEIGH, Lord. XXXI. On the problem of random vibrations, and of random flights in one, two, or three dimensions. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 1919, 220, 321, 1919

determinism from time series **Kaplan-Glass approach**

- *renormalization*: $F^* := \left\langle \frac{L_n^m - R_n^m}{1 - R_n^m} \right\rangle_{n>1}$



Lorenz system
white noise

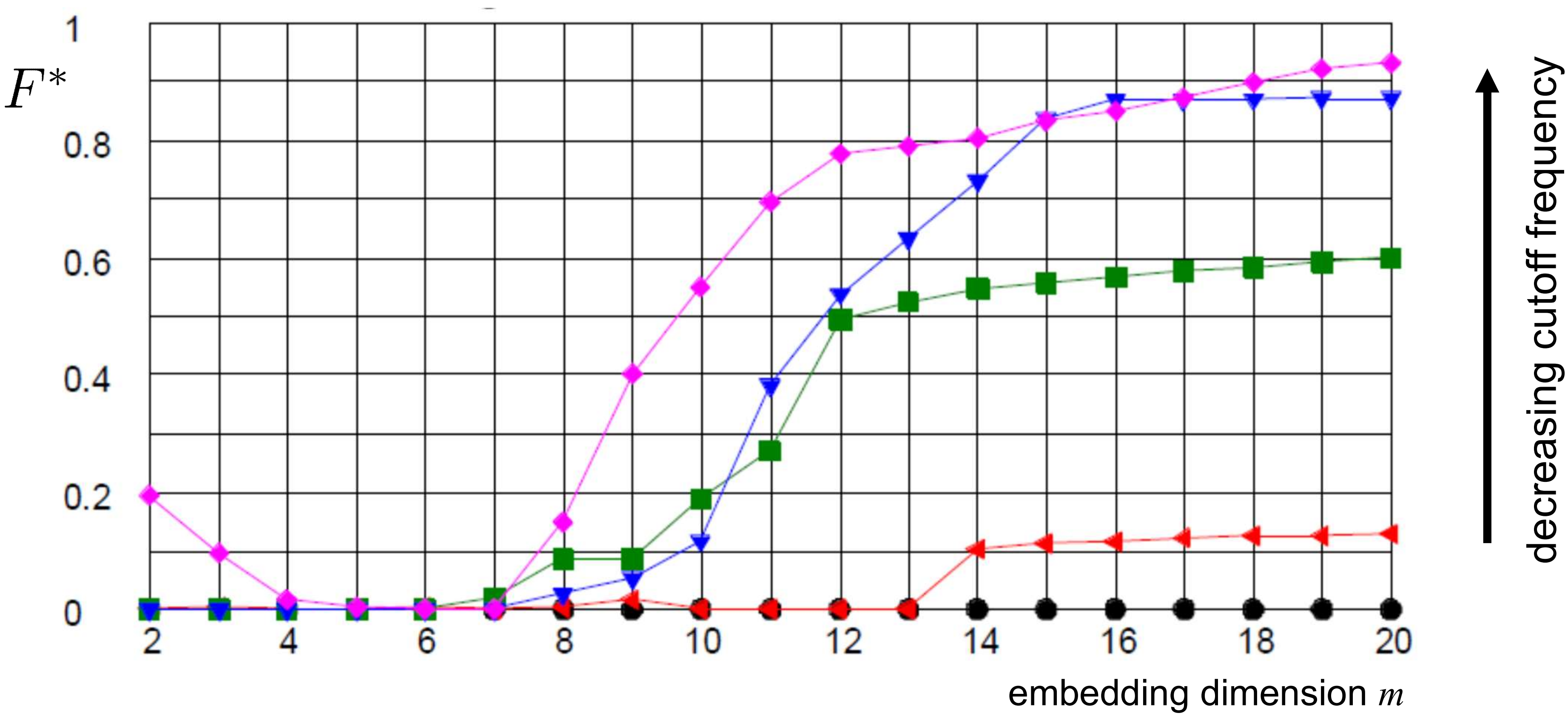
determinism from time series what can go wrong?

field applications

- number of data points (N large enough)
- appropriate embedding (dimension, delay)
- data precision
  adopt to requirement of coarse graining (number of boxes)
- strong correlations in data (sampling interval)
  use Theiler correction (see Dimensions)
- noise, filtering
  filtering can induce determinism
  filtered noise resembles deterministic motion

determinism from time series **what can go wrong?**

insufficient occupation density low-pass filtered noise

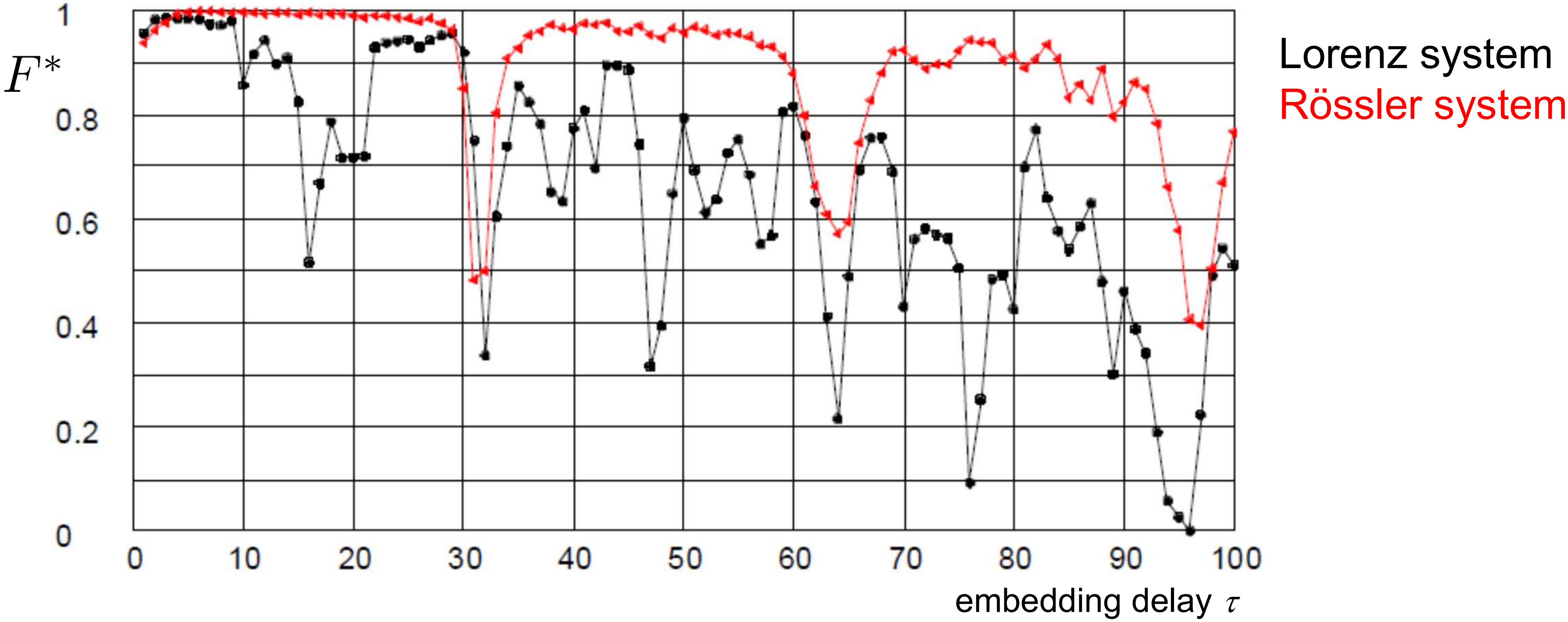


$N = 4096$ data points, fixed embedding delay,
fixed number of boxes/dimension

determinism from time series — **what can go wrong?**

impact of system periodicity — oscillatory systems

minimum at integer multiples of system period length



$N = 4096$ data points, fixed embedding dimension (m=6)

fixed number of boxes/dimension

determinism from time series summary

- easy-to-handle tests for determinism from time series (beware influencing factors)

- useful supplement to “standard” nonlinear analysis techniques

- all methods test for *smoothness* of local flow in phase-space (ε-δ-approach)

- equivalence “smoothness $\Leftrightarrow$ determinism” justified?

- exclusion criterion:
  a stochastic dynamics must yield clearly different findings