Testing for Nonlinearity in Time Series

Null Hypotheses, Surrogates, Monte Carlo Simulation

Brief Recap: Dynamical Invariants and Real-World Data

When analyzing time series from real-world systems

- many prerequisites can not strictly be fulfilled
- limited significance of dynamical invariants
- cannot strictly proof(!) chaos, nonlinearity, deterministic structure
- need to validate assumptions

  (e.g. given a nonlinear system → dynamics also nonlinear?)

⟶ need other methods to
- test for determinism ✓
- test for nonlinearity

testing for nonlinearity **statistical methods**

higher-order moments of amplitude distribution
(see Linear Methods)

Gaussian distribution hints at linearity

deviations may due to:

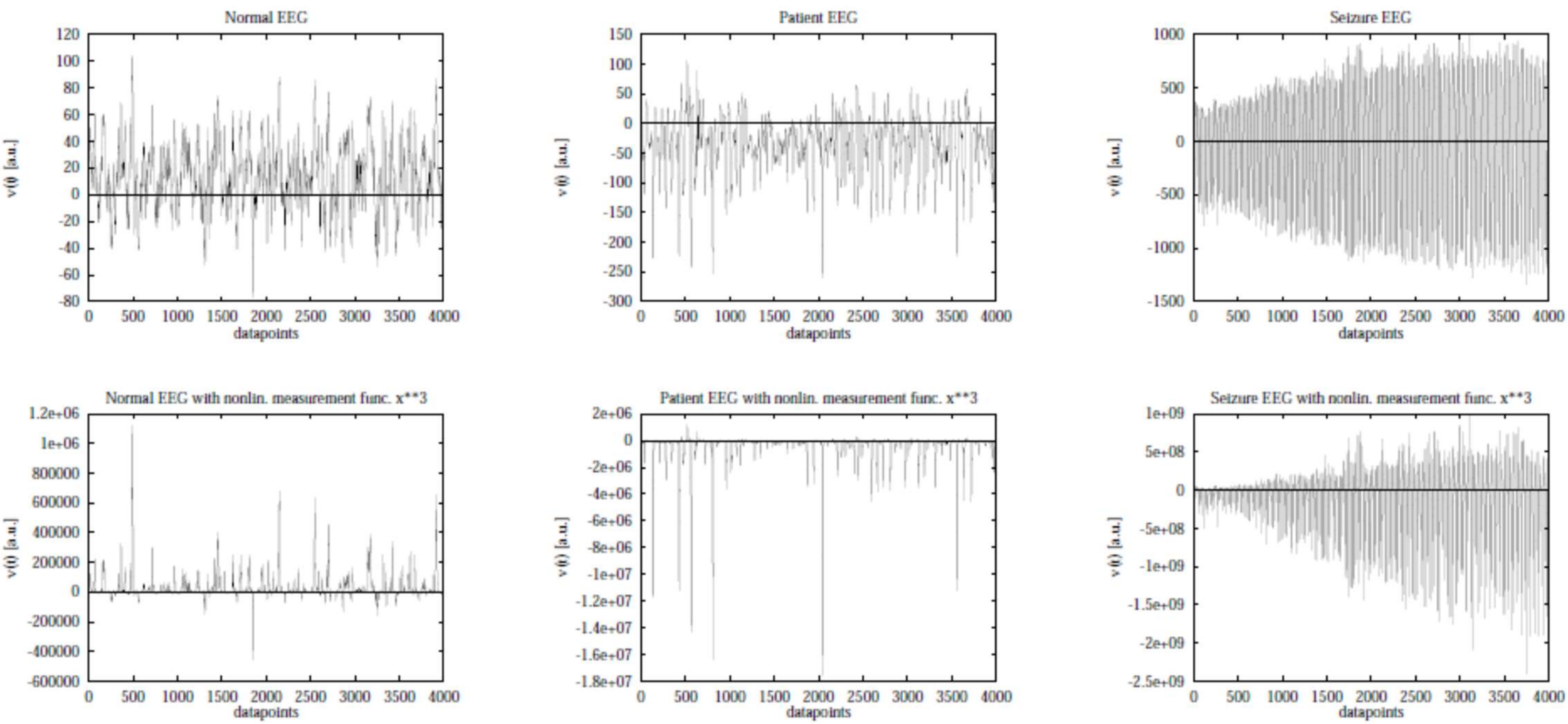
- linear dynamics with non-Gaussian distributed amplitudes (skewness, kurtosis)

- linear dynamics observed with nonlinear measurement function → *static* transformation → *static nonlinearity*

- nonlinear dynamics

testing for nonlinearity statistical methods

impact of nonlinear measurement function



testing for nonlinearity — methods in time domain

decay of autocorrelation
(see Linear Methods)

slow decay hints at long-term correlations in data

deviations may due to:

- non-stationary dynamics
- nonlinear dynamics
- both

indication for nonlinearity if $R_{v^2} \neq (R_v)^2$
caveats:

- autocorrelation of chaotic dynamics decays rapidly to zero
- linear methods cannot distinguish between chaos and noise

testing for nonlinearity **methods in time domain**

time reversibility

regular linear processes are time-reversible

extension (Weiss theorem):
linear Gaussian stochastic processes are time-reversible

caveat: can not invert theorem !

there are time-reversible nonlinear processes
there are time-reversible non-Gaussian linear processes

simple test for “weak” nonlinearity (deviation from time-reversibility):

$$\phi(\delta = 1) := \frac{1}{N-\delta} \sum_{i=\delta+1}^{N} \left(v(i) - v(i-\delta)\right)^3 \stackrel{!}{\neq} 0$$

G. Weiss, Time-Reversibility of Linear Stochastic Processes. J. App. Prob, 12, 831, 1975
T. Schreiber and A. Schmitz, Discrimination power of measures for nonlinearity in a time series, Phys. Rev. E 55, 5443, 1997

testing for nonlinearity **methods in time domain**

time reversibility

$$\phi(\delta = 1) := \frac{1}{N-\delta} \sum_{i=\delta+1}^{N} \left(v(i) - v(i-\delta)\right)^3 \stackrel{!}{\neq} 0$$

some examples:

white noise	$\phi(1) = 0.00$
Hénon system	$\phi(1) = -0.34$
Lorenz system	$\phi(1) = -0.002$
normal EEG	$\phi(1) = -0.17$
patient EEG	$\phi(1) = -6.24$
seizure EEG	$\phi(1) = -214.02$

nonlinear
or
non-stationary?

testing for nonlinearity

statistical methods and methods in time domain:

- mostly “static” properties given by amplitude distribution
- time reversibility not unique
- interpretability
- poor discriminatory power

testing for nonlinearity **methods in phase space**

time reversibility:

"a time series $v(t)$ is time-reversible, if the probability density function $\rho(\mathbf{v})$ of phase-space vectors is invariant under time reversal for all embedding dimensions and delays"

reconstruct phase space:

$$\left(v_i, v_{i-\tau}, v_{i-2\tau}, \ldots, v_{i-(m-1)\tau}\right)$$

show invariance with

$$\rho(\mathbf{v}) \stackrel{!}{=} \rho(U^{-1}\mathbf{v})$$

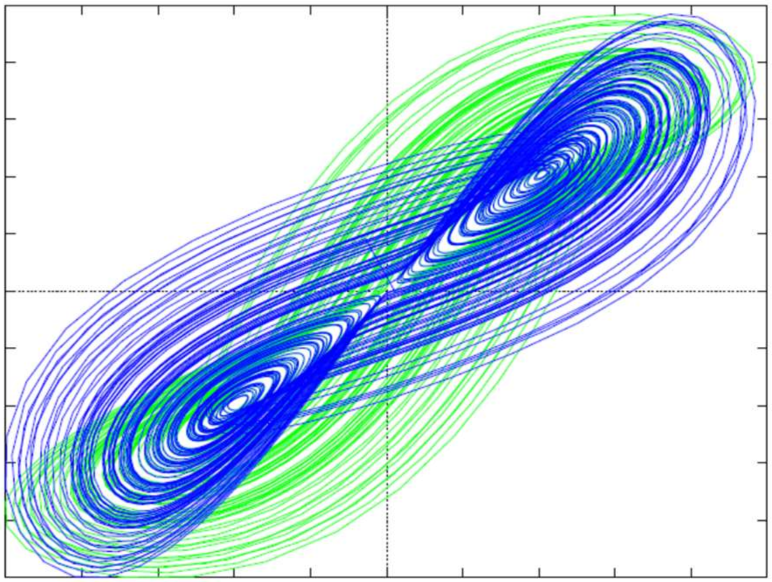
where

$$U^{-1} := \left(v_i, v_{i-\tau}, \ldots, v_{i-(m-1)\tau}\right) \to \left(v_{i-(m-1)\tau}, \ldots, v_{i-\tau}, v_i\right)$$

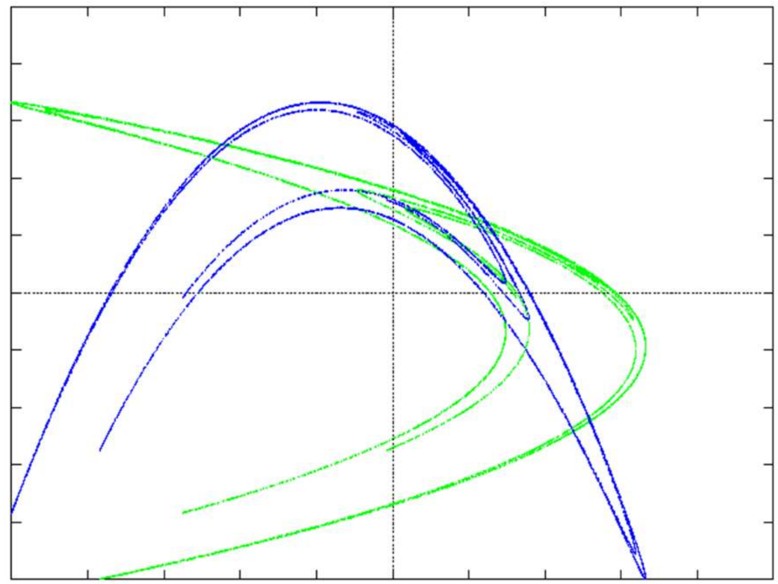
C. Diks et al., Phys Rev E 53, 2169, 1996; M. van der Heyden et al., Phys Lett A 216, 283, 1996

testing for nonlinearity **methods in phase space**

original and time-reversed attractor



Lorenz system



Hénon system

requires sensitive tests to prove invariance

testing for nonlinearity **methods in phase space**

BDS test:

initially used to improve predictability of stock prices

method is based on correlation sum:

$$C_2(\epsilon, m) := \frac{1}{N}\sum_i \left(\frac{1}{N}\sum_j \Theta\left(\epsilon - |\vec{v_i} - \vec{v_j}|\right)\right)$$

for uncorrelated (**i**ndependent **i**rregularly **d**istributed) data, we have:

$$C_2(\epsilon, m) = [C_2(\epsilon, m = 1)]^m$$

any kind of (linear/nonlinear) correlation implies:

$$\frac{C_2(\epsilon, m)}{[C_2(\epsilon, m=1)]^m} > 1$$

W.A. Brock, W.D. Dechert, J.A. Scheinkman & B. LeBaron (1988), Dept.of Economics, University of Wisconsin Press, Madison

testing for nonlinearity **methods in phase space**

BDS test:

test is not applied to original time series, since correlations are known trivially (power spectrum, autocorrelation)

instead, find optimal AR(k) model for time series

$$\epsilon_i = v_i - \sum_{j=1}^{k} \alpha_j v_{i-j}$$

if model describes the data "sufficiently well"

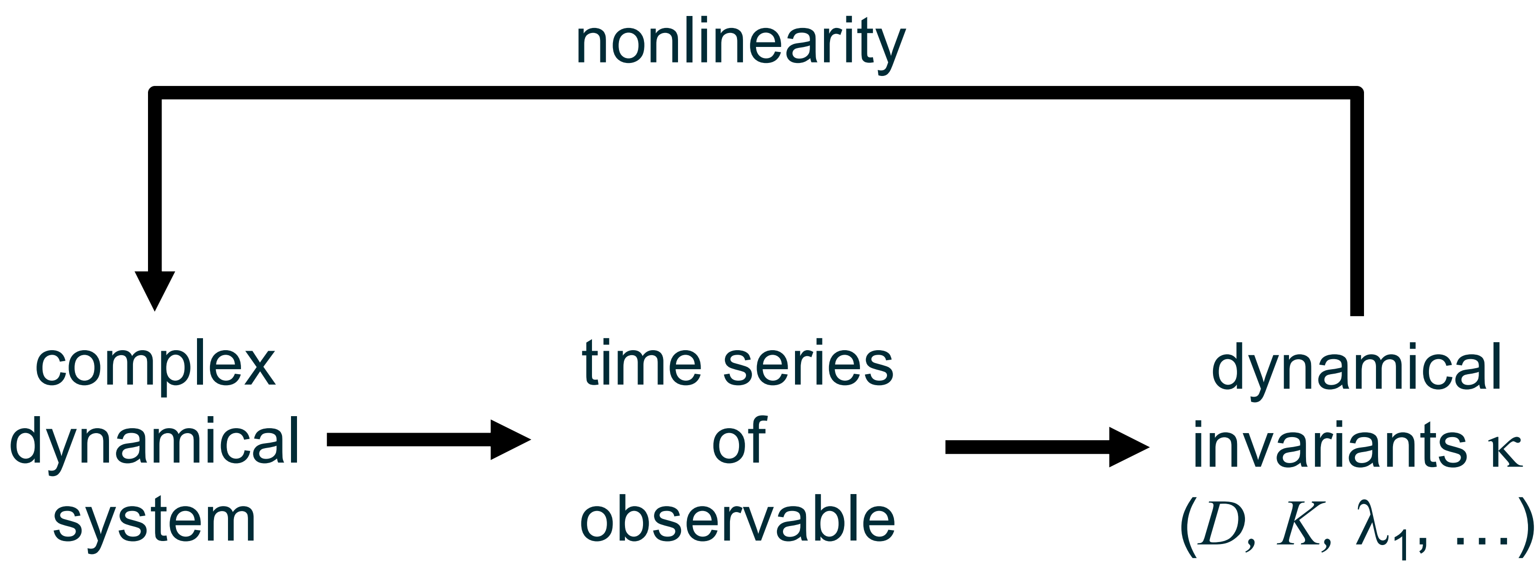
- "innovations" ε_i are independent (i.i.d.)
- BDS test positive
- hint for linearity

testing for nonlinearity

methods in phase space:

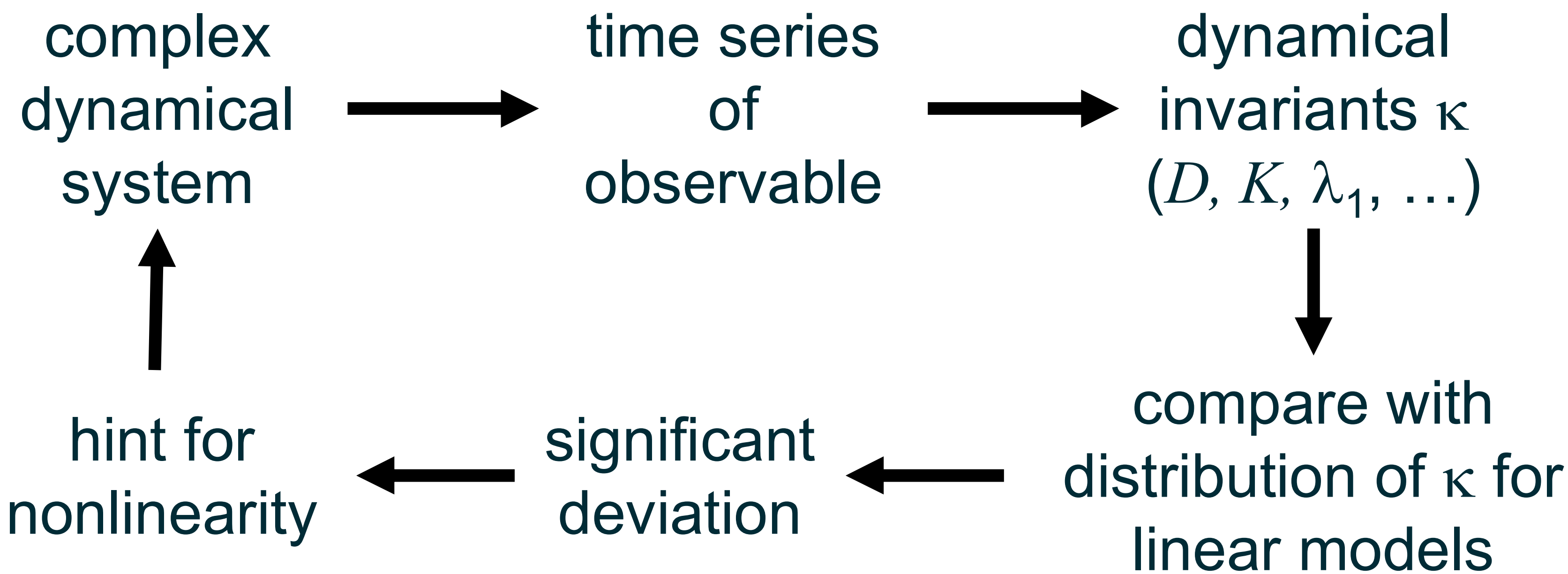
- mostly “static” properties given by probability density function
- time reversibility not unique
- interpretability (binary decision (y/n) only)
- poor discriminatory power
- do not allow one to judge whether ***dynamical invariants*** characterize ***nonlinear deterministic structures***

testing for nonlinearity



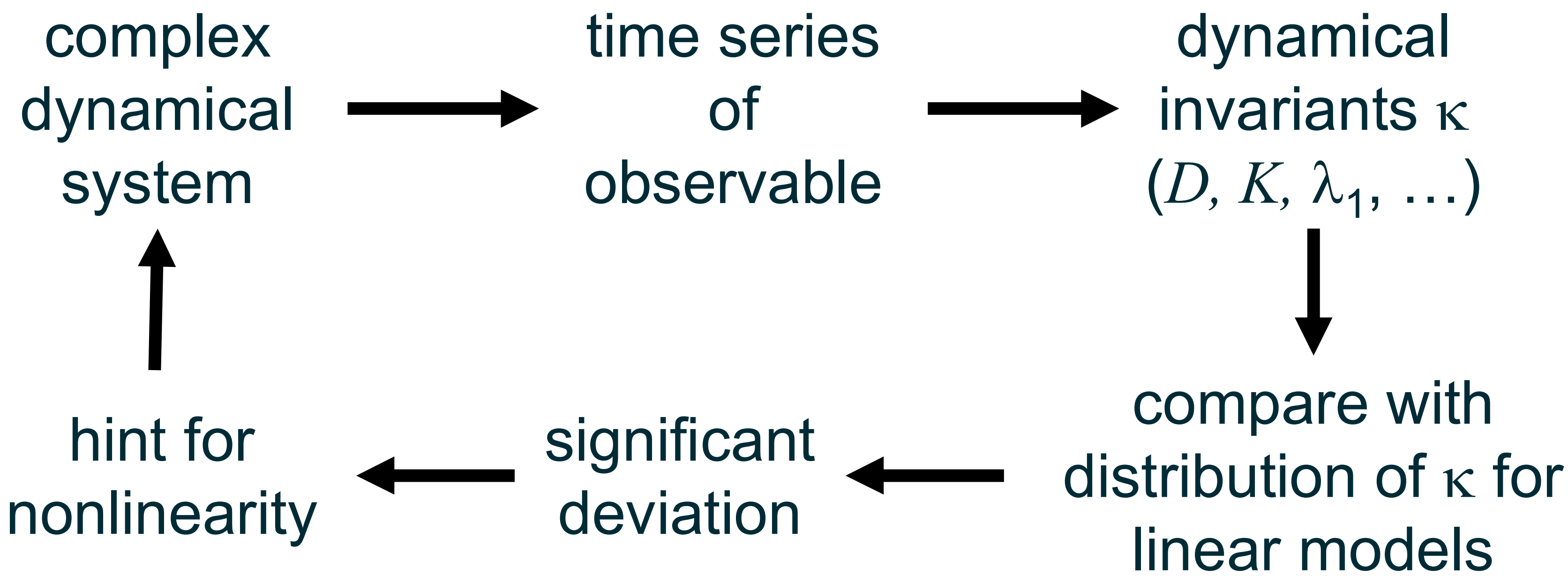
is this reliable ?

testing for nonlinearity — a more reliable ansatz



T. Schreiber and A. Schmitz, Surrogate Time Series, Physica D 142, 346, 2000

testing for nonlinearity — a more reliable ansatz



problems:

- distribution of κ for linear models a priori unknown
- only ***one*** time series (finite, limited precision, …)

T. Schreiber and A. Schmitz, Surrogate Time Series, Physica D 142, 346, 2000

testing for nonlinearity with surrogates

approaching the problem with hypothesis testing:

1. define appropriate null hypothesis:
   e.g. **“the data have been generated by some linear, stochastic, Gaussian, stationary process”**

2. build ***surrogate*** time series that have the ***same statistical properties*** as the original time series ***except nonlinearity***

3. estimate dynamical invariant for original time series and a surrogate ensemble

4. apply robust test statistics to reject / confirm null hypothesis

T. Schreiber and A. Schmitz, Surrogate Time Series, Physica D 142, 346, 2000

testing for nonlinearity with surrogates

building surrogate time series:

typical realizations

- require explicit model equations
- derive model parameters from time series
- build surrogates via Monte-Carlo realizations

constrained realizations (bootstrapping)

- phase randomization (FT)
- amplitude-adjusted phase randomization (AAFT)
- iteratively amplitude-adjusted phase randomization (IAAFT)
- random shuffling (RS)

testing for nonlinearity with surrogates

building FT surrogate time series

Idea: dynamical nonlinearity associated with phase changes

null hypothesis:
the data have been generated by a linear, stochastic, stationary process with Gaussian distributed amplitudes and possibly observed through some (static) nonlinear measurement function

approach:

(1) Fourier-transform of original time series

(2) replace phases by random numbers from $[0,2\pi)$
(preserves amplitude spectrum)

(3) inverse Fourier-transform

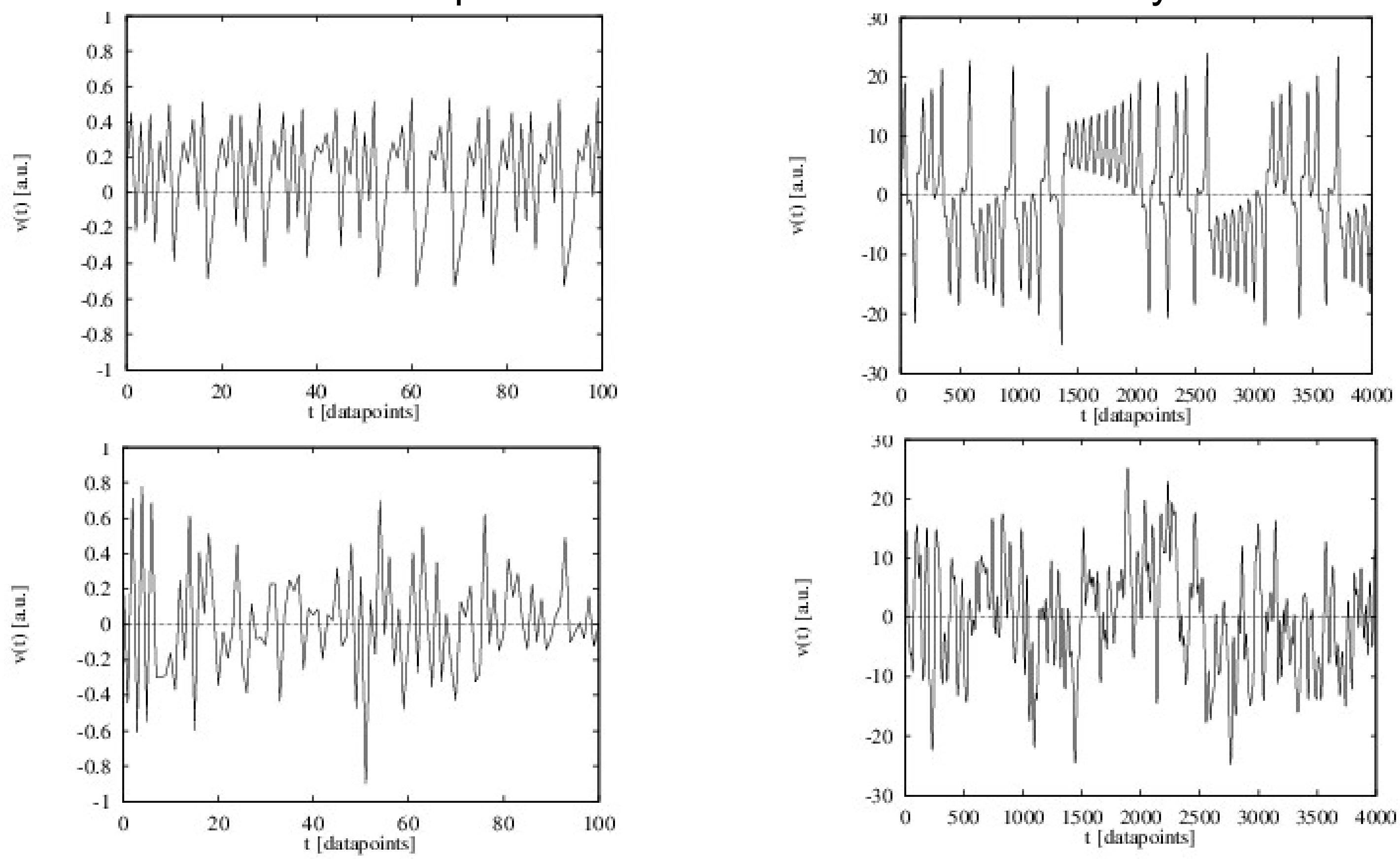
Pijn, J. P. Diss Amsterdam 1991;
Theiler, J., Eubank, S., Longtin, A., Galdrikian, B. and Farmer, J.D. Testing for nonlinearity in time series: the method of surrogate data. *Physica D*, *58*, 77, 1992

testing for nonlinearity with surrogates

building FT surrogate time series examples

Hénon map — Lorenz system

original time series / FT surrogate

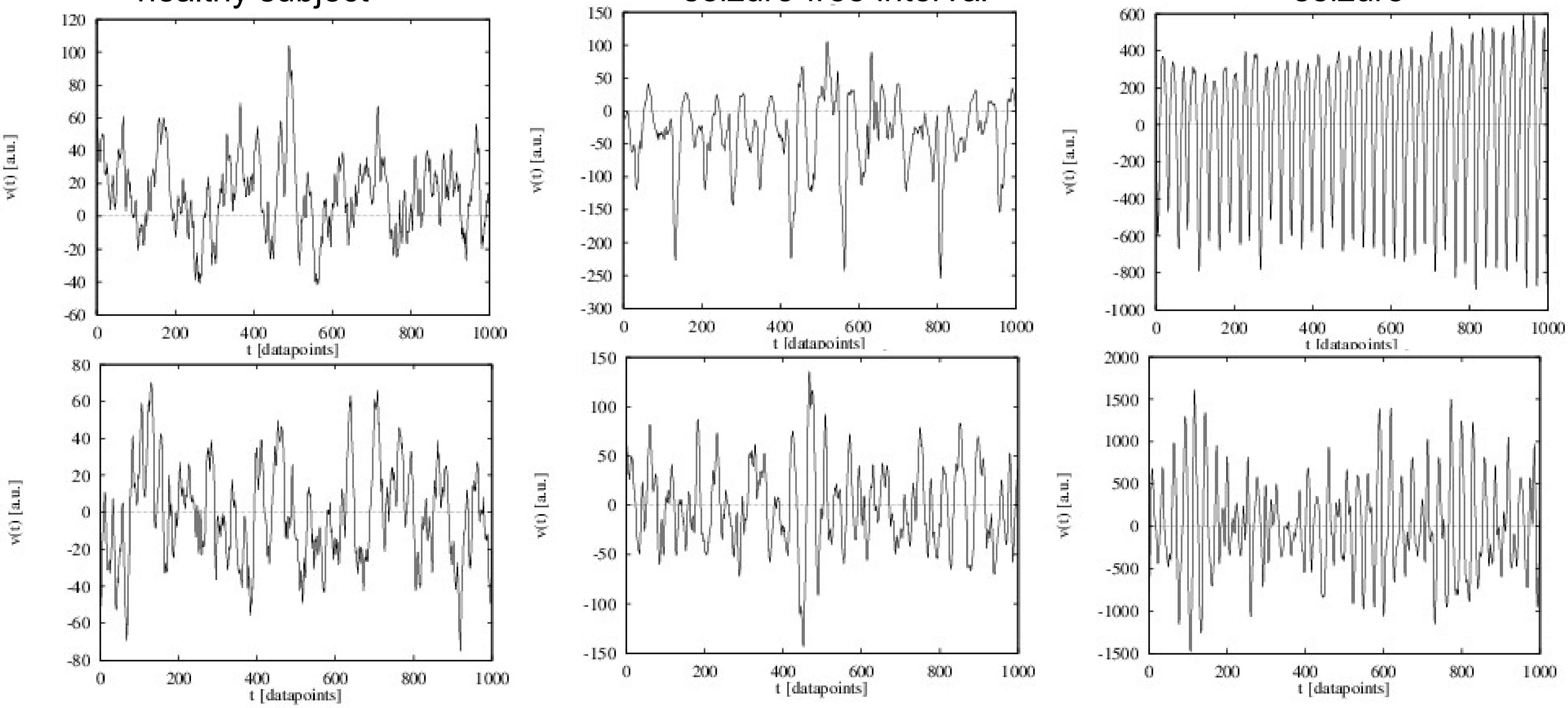


testing for nonlinearity with surrogates

building FT surrogate time series: examples

healthy subject | epilepsy patient seizure-free interval | epilepsy patient seizure

original time series / FT surrogate



testing for nonlinearity with surrogates

building FT surrogate time series: *summary*

properties of phase-randomized surrogates:

- amplitude distribution and Fourier spectra of original time series and surrogate time series identical
  (with Wiener-Khinchine theorem: identical autocorrelation functions)
- means and standard deviations of amplitude distributions identical
- shape of distributions may vary

caveat: - FT surrogates have Gaussian distributed amplitudes!
- can lead to false detections of nonlinearity!

testing for nonlinearity with surrogates

building AAFT surrogate time series:

Idea: dynamical nonlinearity associated with phase changes

null hypothesis:
the data have been generated by a linear, stochastic, stationary process with arbitrary amplitude distribution and possibly observed through some (static) nonlinear measurement function

approach:

(1) generate sequence of Gaussian distributed random numbers

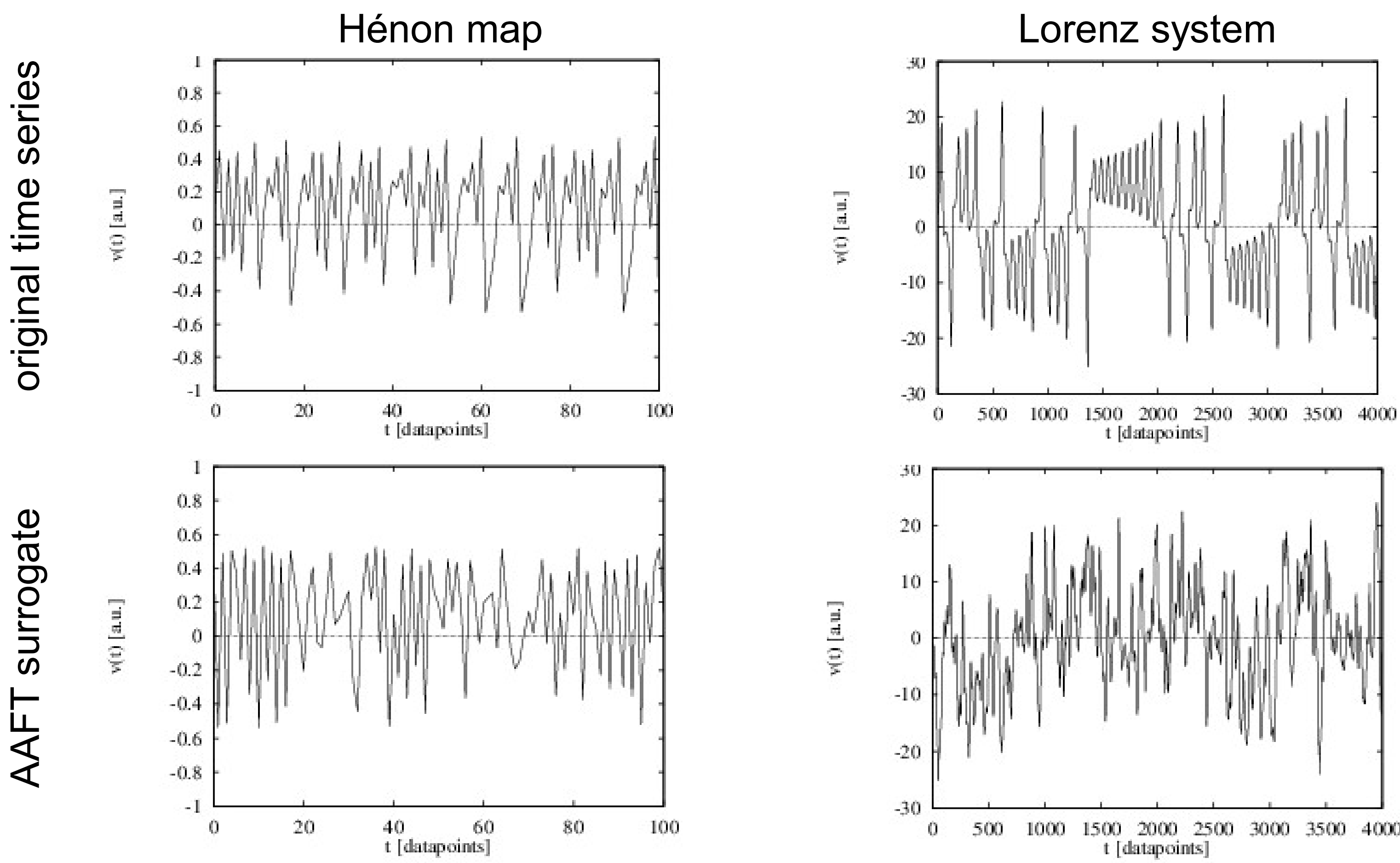
(2) generate FT surrogate of that sequence

(3) rescale amplitudes of surrogate using the rank-ordering of amplitudes of original time series

J. Theiler; D. Prichard "Constrained-realization Monte-Carlo method for hypothesis testing". Physica D. 94, 221, 1996.

testing for nonlinearity with surrogates

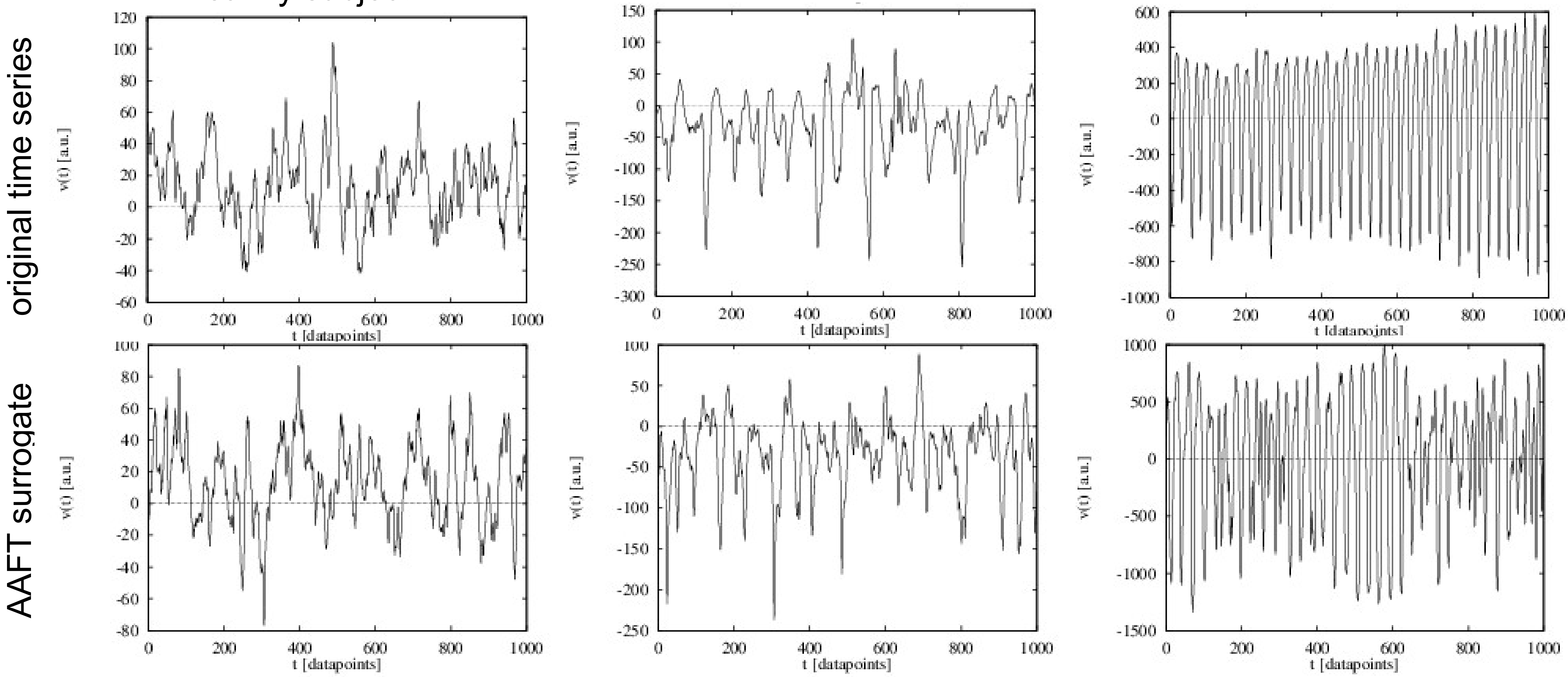
building AAFT surrogate time series examples



testing for nonlinearity with surrogates

building AAFT surrogate time series: examples

healthy subject | epilepsy patient seizure-free interval | epilepsy patient seizure



testing for nonlinearity with surrogates

building AAFT surrogate time series: *summary*

properties of amplitude-adjusted phase-randomized surrogates:

- same as FT surrogates but amplitude distribution optimally adjusted to the one of original time series
- Fourier spectrum optimally adjusted for $N \to \infty$ and weakly correlated data only

caveat: - Fourier spectrum too flat (almost white) in case of finite data and strong correlations!
- can lead to false detections of nonlinearity!

T. Schreiber, A. Schmitz / Physica D 142 (2000) 346–382

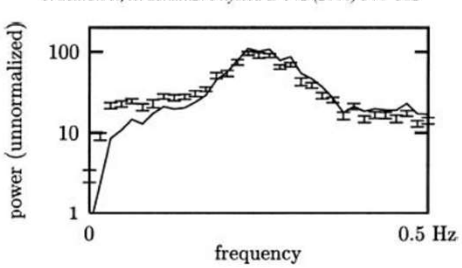


Fig. 3. Discrepancy of the power spectra of human breath rate data (solid line) and 19 AAFT surrogates (dashed lines). Here the power spectra have been computed with a square window of length 64.

testing for nonlinearity with surrogates

building IAAFT surrogate time series:

Idea: dynamical nonlinearity associated with phase changes

null hypothesis:
the data have been generated by a linear, stochastic, stationary process with arbitrary amplitude distribution and possibly observed through some (static) nonlinear measurement function

approach:

(1) iterative version of AAFT

(2) repeat until Fourier spectrum optimally adjusted;
stopping criterion after i iterations, e.g.:

$$\delta^{(i)} = \sum_{k=1}^{K_{\text{Nyq}}} \frac{(\mathcal{S}_{\text{surr}}(k) - \mathcal{S}_{orig}(k))^2}{\mathcal{S}_{orig}(k)^2} \stackrel{!}{=} \min$$

T. Schreiber; A. Schmitz. "Improved Surrogate Data for Nonlinearity Tests". Phys. Rev. Lett. 77, 635, 1996

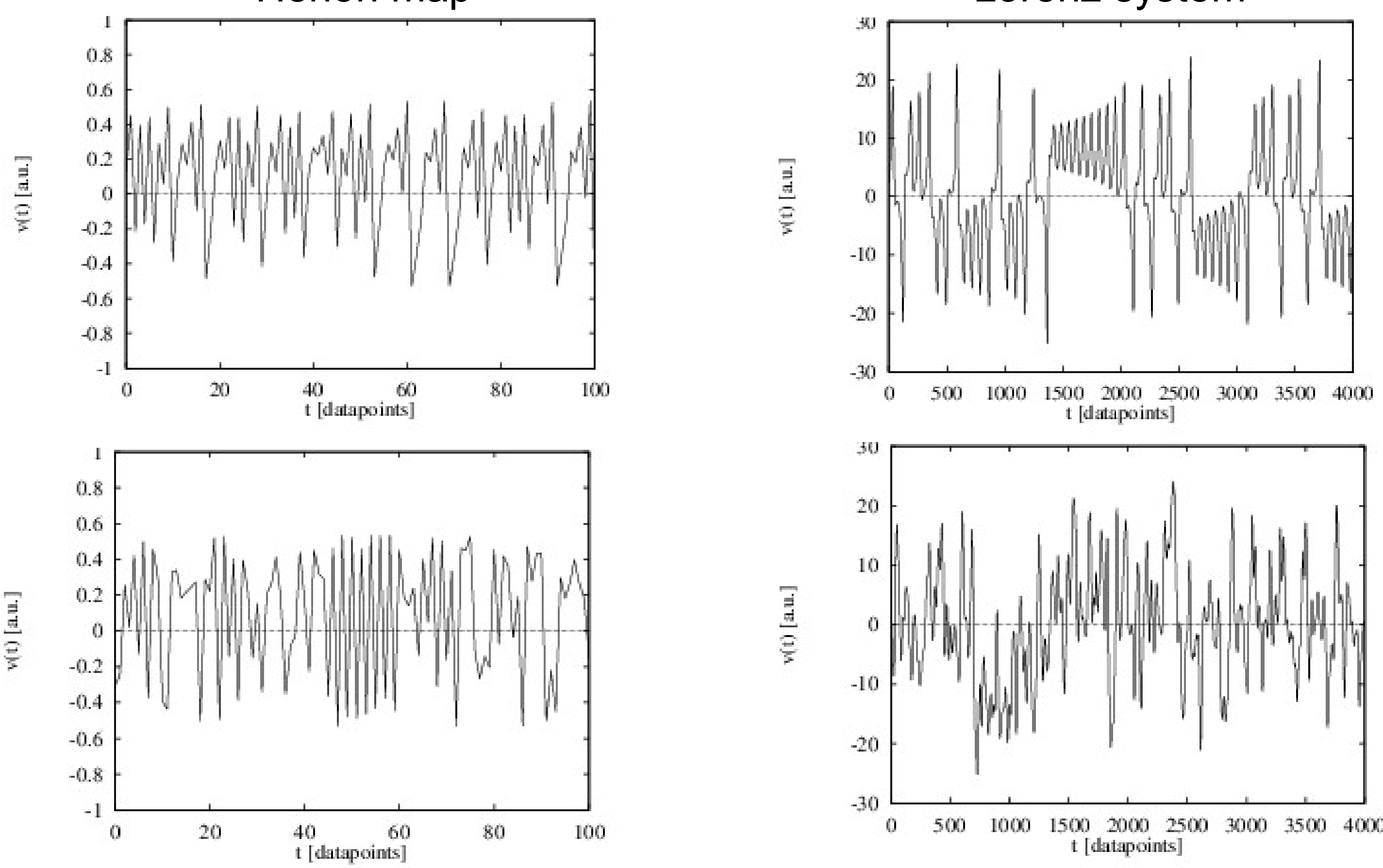
testing for nonlinearity with surrogates

building IAAFT surrogate time series examples

Hénon map | Lorenz system

original time series

IAAFT surrogate



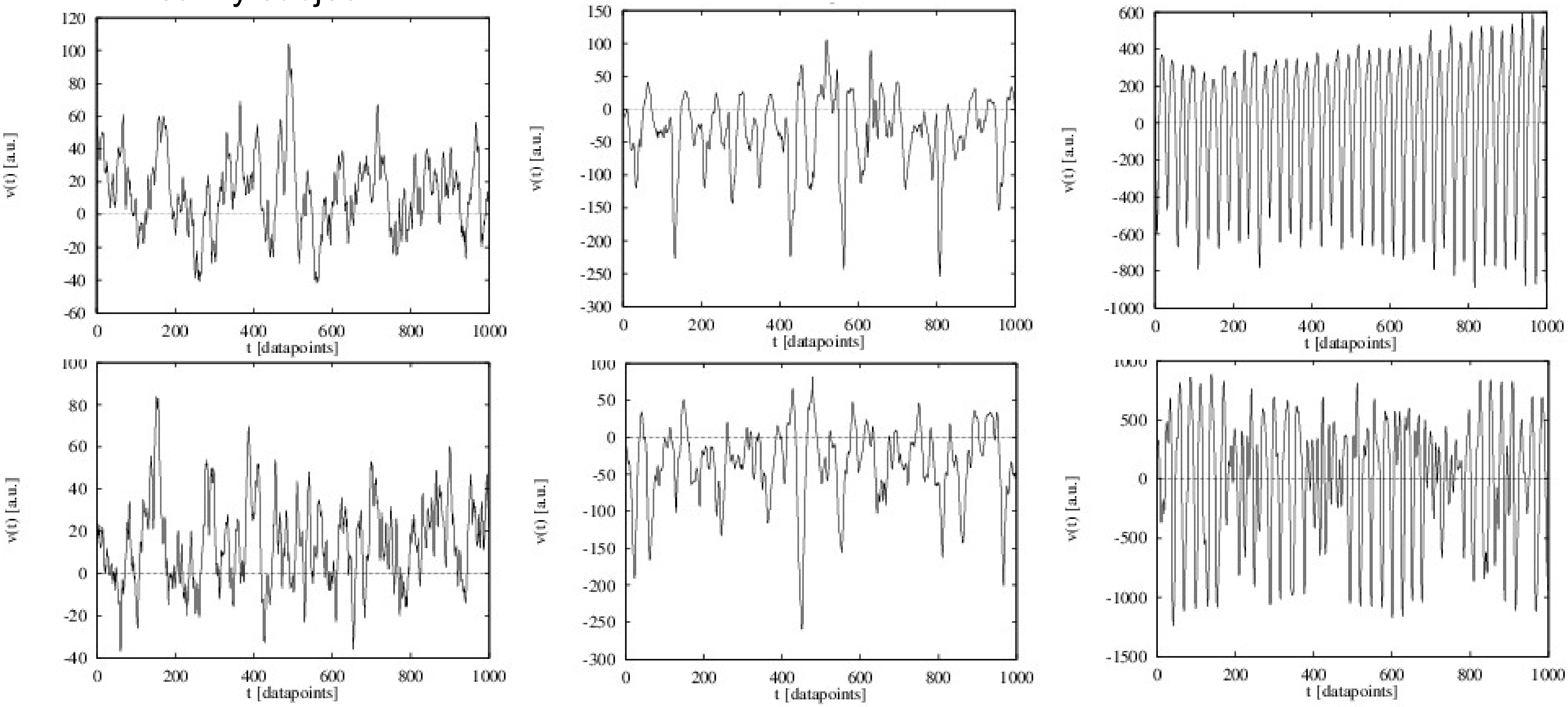
testing for nonlinearity with surrogates

building IAAFT surrogate time series: examples

healthy subject | epilepsy patient seizure-free interval | epilepsy patient seizure

original time series

IAAFT surrogate



testing for nonlinearity with surrogates

building IAAFT surrogate time series: *summary*

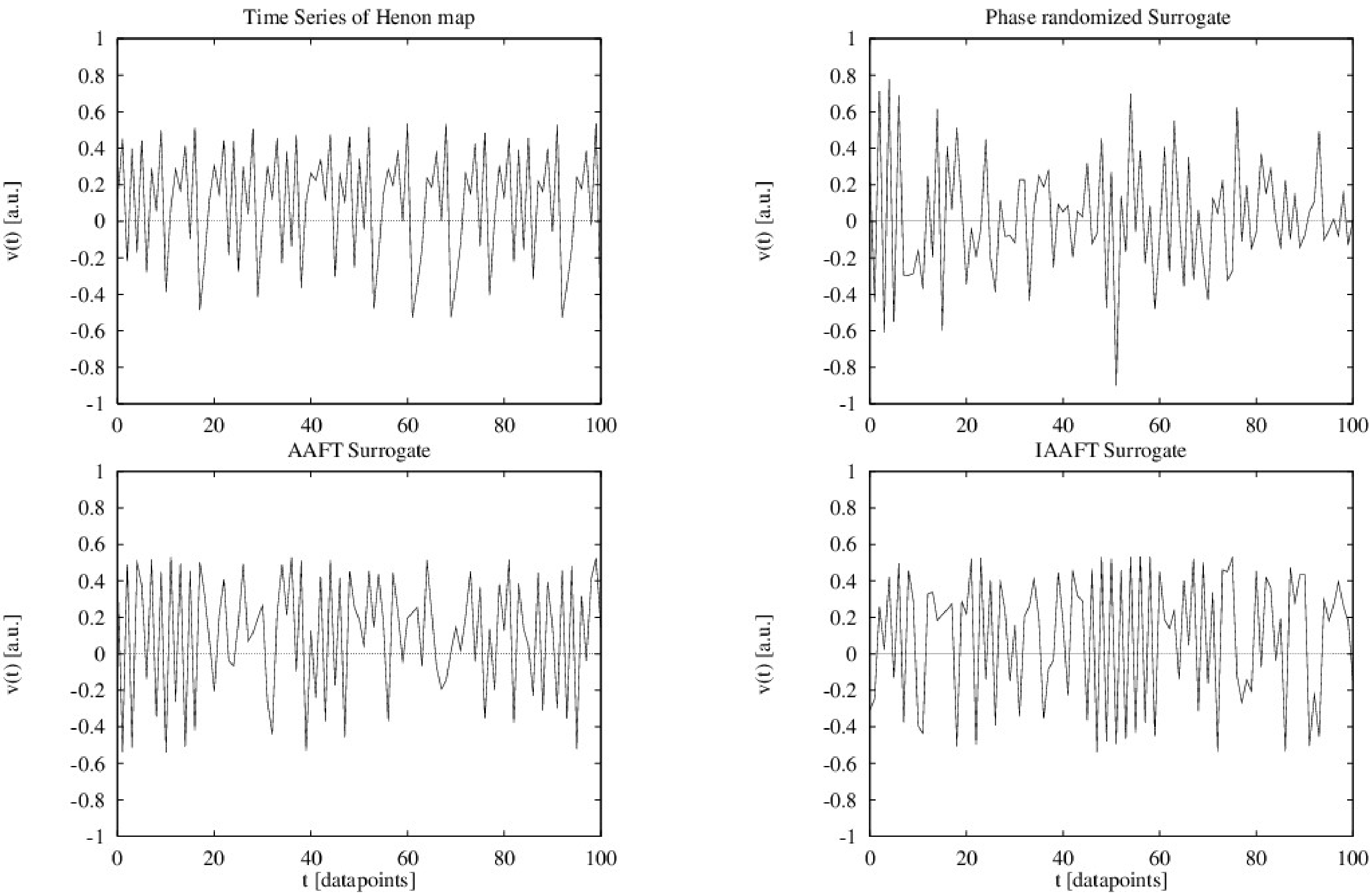
properties of iterative amplitude-adjusted phase-randomized surrogates:

- same as AAFT surrogates but amplitude distribution and Fourier spectrum optimally adjusted to the ones of original time series
- note: Fourier spectrum optimally adjusted for $N \to \infty$

caveat: avoid over-iteration!

testing for nonlinearity with surrogates

comparing FT, AAFT, IAFFT surrogate time series:



testing for nonlinearity with surrogates

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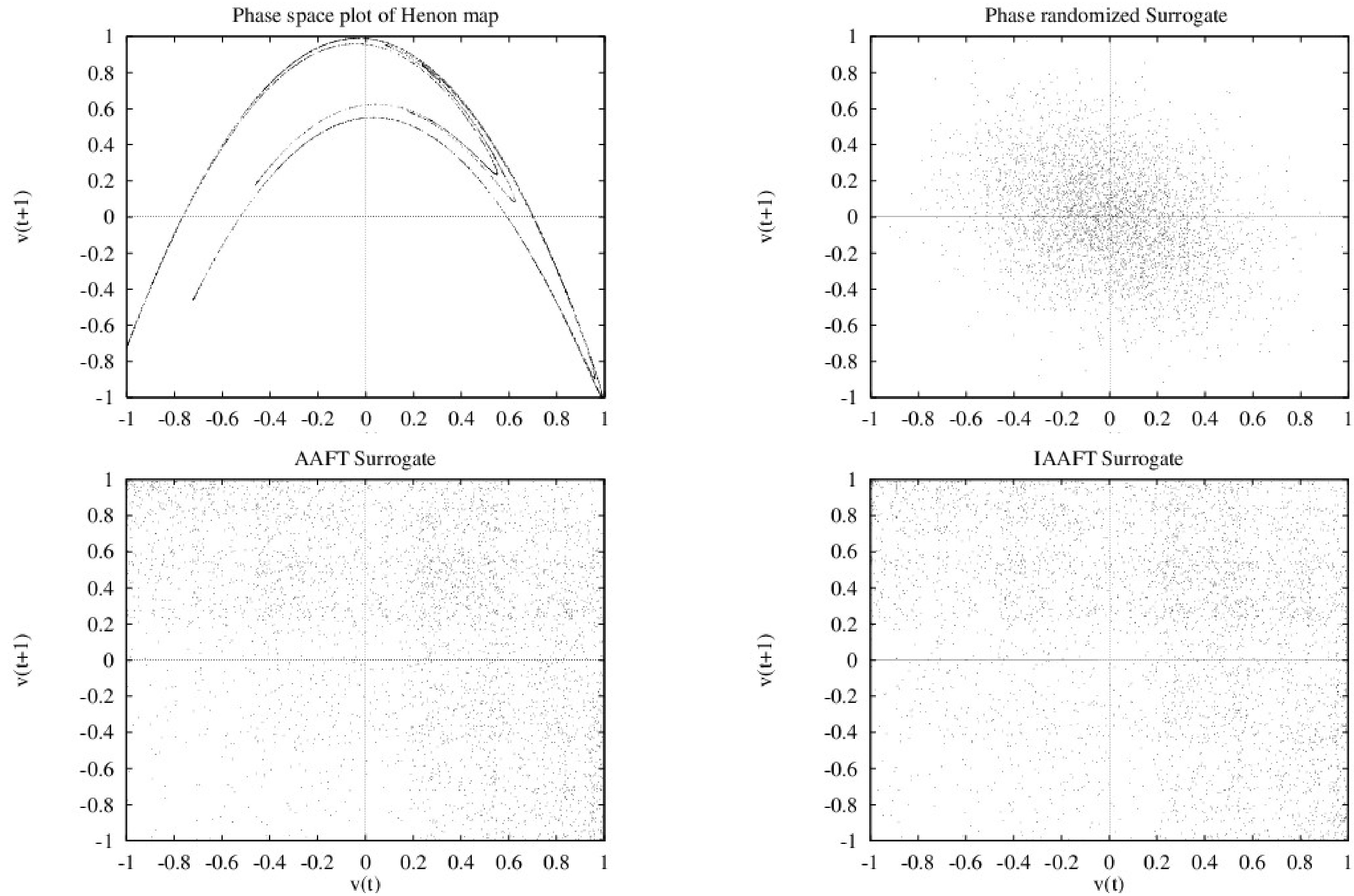

testing for nonlinearity with surrogates

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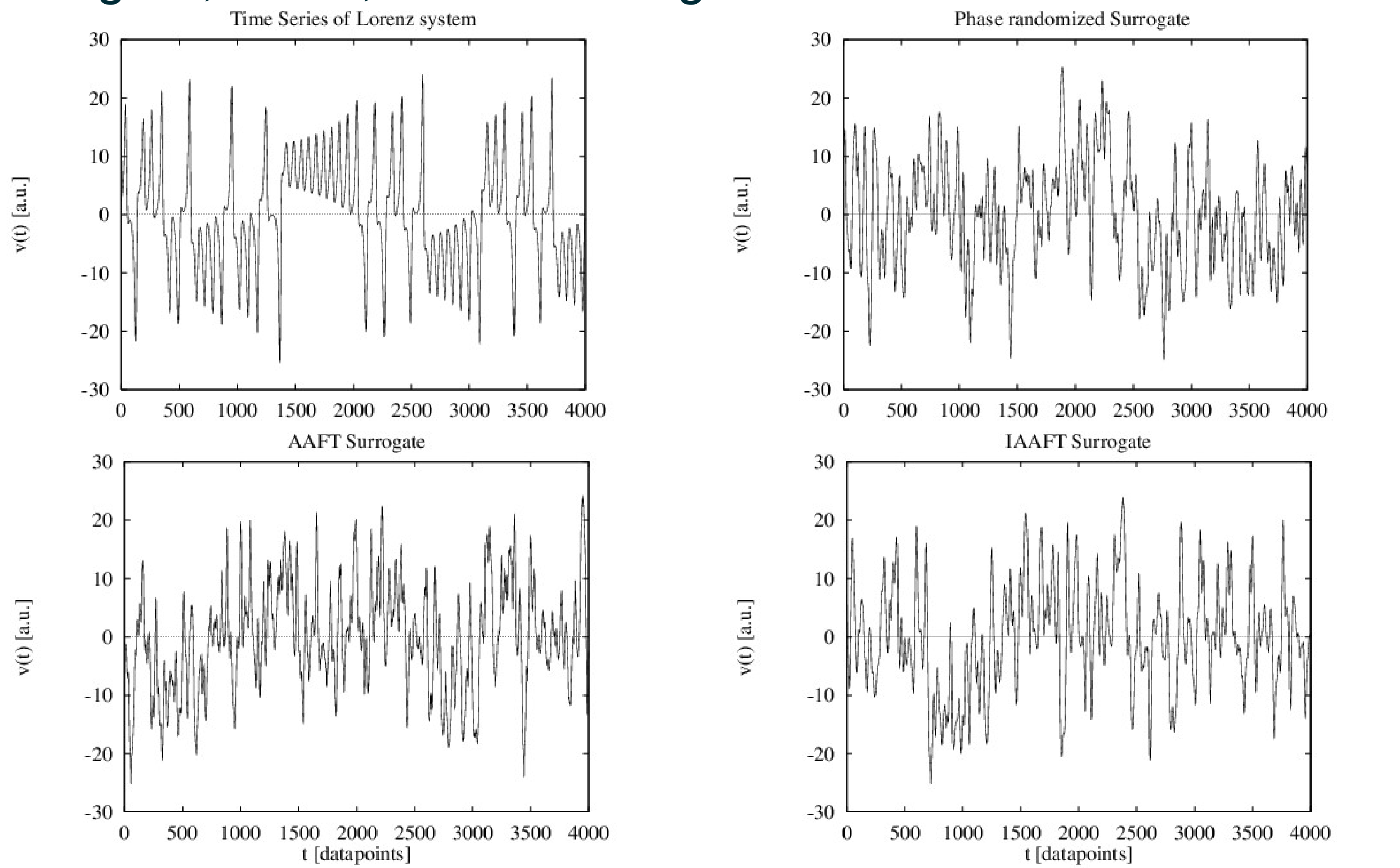
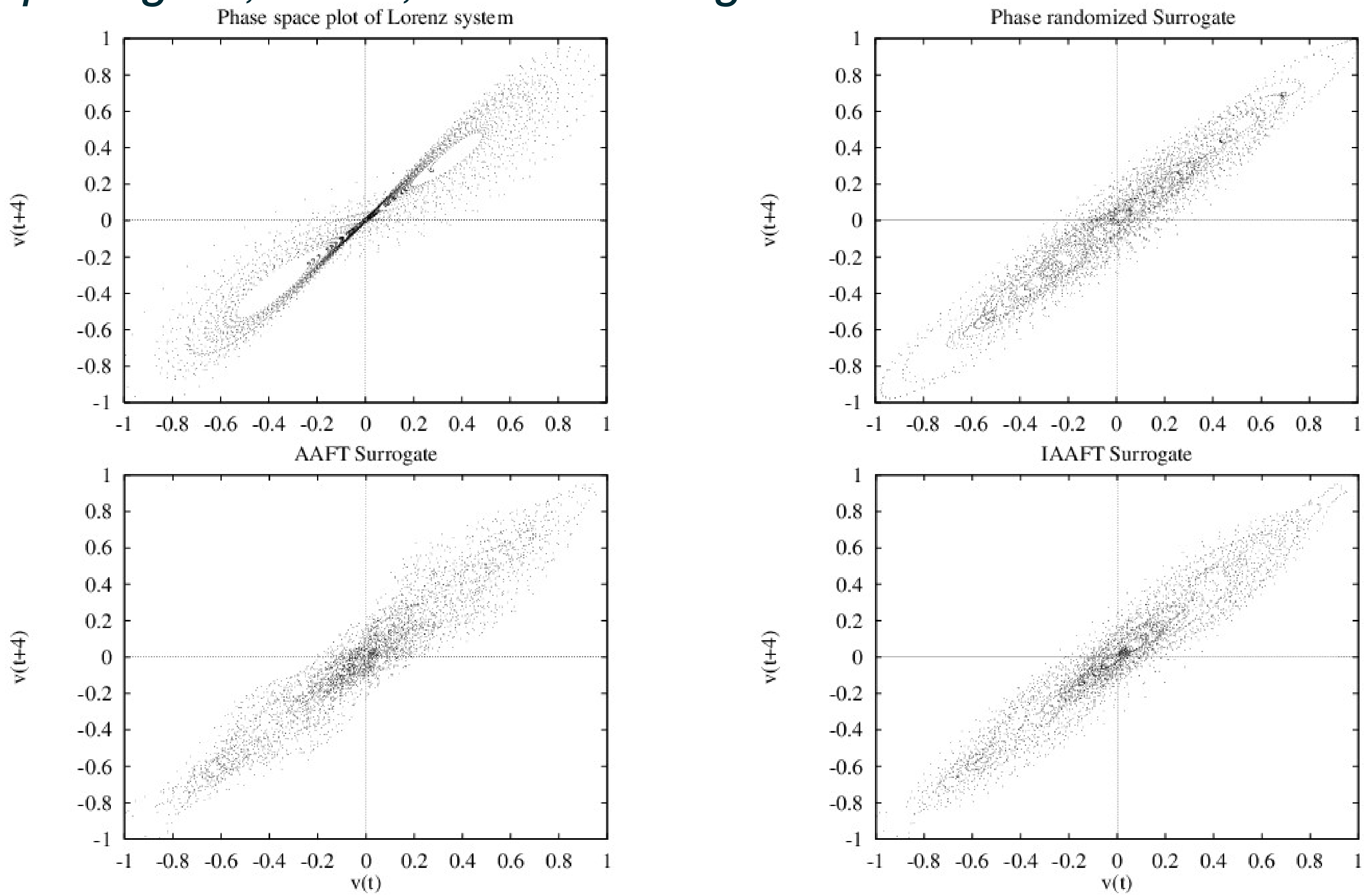


testing for nonlinearity with surrogates

comparing FT, AAFT, IAFFT surrogate time series:



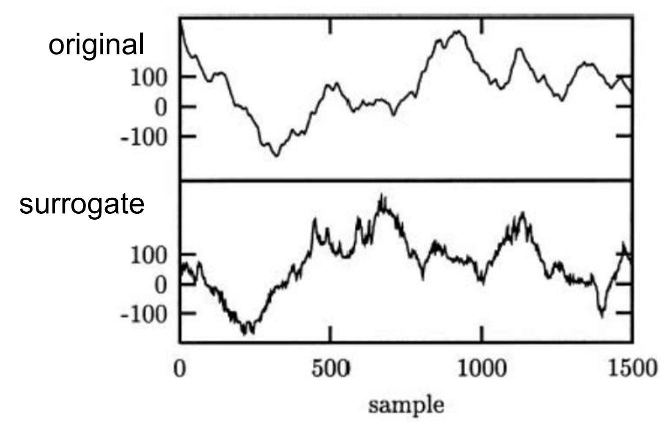
testing for nonlinearity with surrogates

things to bear in mind when generating FT, AAFT, IAFFT surrogate time series:

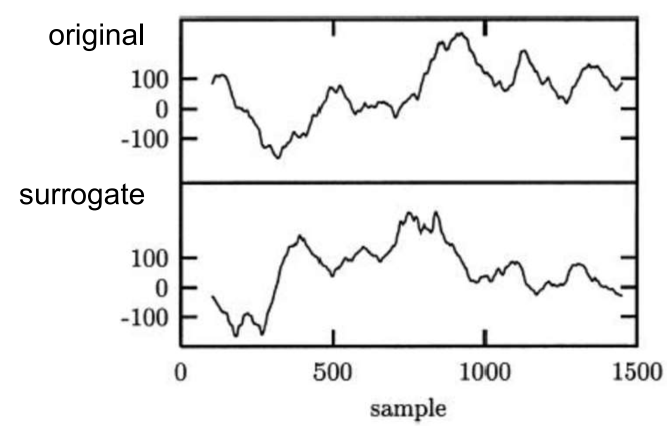
"end-point-mismatch" (or edge effect):

- need continuous time series for Fourier transform
- usually employ *windowing* or *zero-padding* in case of finite data
- can't use this here; need invertibility of Fourier transform!
- end-point-mismatch induces a more white Fourier spectrum
- possible ansatz: shift time series until mismatch minimized

with end-point-mismatch



end-point-mismatch minimized



T. Schreiber and A. Schmitz, Surrogate Time Series, Physica D 142, 346, 2000

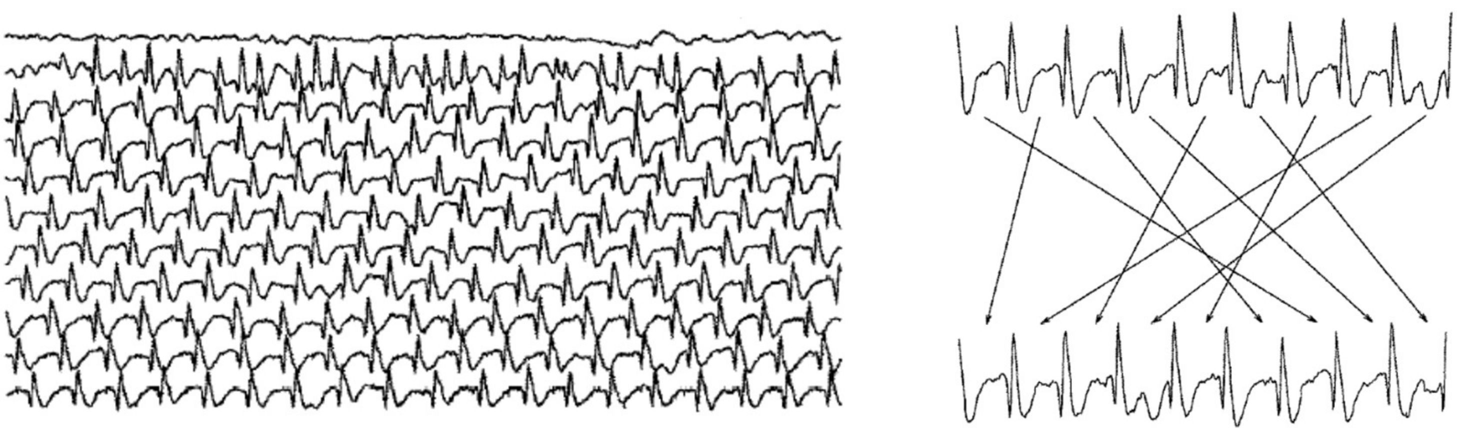
testing for nonlinearity with surrogates

building surrogate time series: *random shuffling*

null hypothesis:
no nonlinear deterministic structure in variability of periodicities

ansatz:
identify recurring patterns and perform random shuffling

field of applications: near-periodic data
(e.g. sun spots, seizure EEG, electrocardiogram (ECG))



from J. Theiler On the evidence for how-dimensional chaos in an epileptic electroencephalogram. Phys. Lett. A 196, 335, 1995

testing for nonlinearity with surrogates

other methods to build surrogate time series:

- use wavelet transform instead of Fourier transform (other basis function; more localized in time- and frequency domain)
- include trends in null hypothesis
- include non-stationarity in null hypothesis
- extensions to more general linear processes
- testing separately for dynamical and static nonlinearities (avoid non-detections of nonlinearities)
- surrogates for sparsely quantized time series
- multifractal surrogates
- iterative digitally filtered shuffled surrogates
- …

G. Lancaster et al. Surrogate data for hypothesis testing of physical systems. Physics Reports 748, 1, 2018

testing for nonlinearity with surrogates

rejecting/accepting null hypothesis (*non-parametric vs parametric tests*)

how to design a test and how many surrogates do you need?

parametric test (probably most intuitive and easy, but **not recommendable**!)

- estimate dynamical invariant κ for original time series (O) and a N_s surrogates (S)
- estimate mean and variance of κ for surrogates and estimate significance (s) of deviation by "*number of sigmas*": $s = \frac{\bar{\kappa}_S - \kappa_O}{\sigma_{\kappa_S}}$
- caveat: this test assumes κ to be Gaussian distributed and does not provide you with an estimate of the number N_s of surrogates

testing for nonlinearity with surrogates

rejecting/accepting null hypothesis (*non-parametric vs parametric tests*)

how to design a test and how many surrogates do you need?

*non-parametric (**robust**) test*

- chose probability α to falsely reject null hypothesis (e.g., 0.05)
  significance level $p = 1 - \alpha$
- estimate dynamical invariant κ for original time series (O) and N_s surrogates (S)
- *one-sided test* (e.g., only a given value of κ is of interest)
  generate $N_\mathrm{s} = (1/\alpha - 1)$ surrogates
  α is probability that original data take a given value of κ by chance
- *two-sided test* (e.g., min or max value of κ is of interest)
  generate $N_\mathrm{s} = (2/\alpha - 1)$ surrogates
  α is probability that original data take either min or max of κ by chance

robustly testing for nonlinearity with surrogates

choose significance level

$p = (1-\alpha) \times 100\%$

$N_s = (1/\alpha - 1)$ surrogates (one-sided)

$N_s = (2/\alpha - 1)$ surrogates (two-sided)

(e.g., $N_s = 19$ resp. 39 for $p = 95\ \%$

← original time series

dynamical invariant κ differs for original and surrogates

↓

reject null hypothesis: indication for nonlinearity

dynamical invariant κ does not differs for original and surrogates

↓

accept null hypothesis: no indication for nonlinearity

robustly testing for nonlinearity with surrogates

interpretation:

- statistical test only, it is not a proof for nonlinearity!
- rejection of null hypothesis only provides necessary but not sufficient condition for dynamical invariant to indicate nonlinearity
- acceptance of null hypothesis does not indicate its correctness
- whenever a null hypothesis is rejected, it is always very important to keep in mind that the complementary hypothesis is very comprehensive and might include many different reasons that are possibly responsible for this rejection
- consider including other (statistical) properties of your time series into the null hypothesis

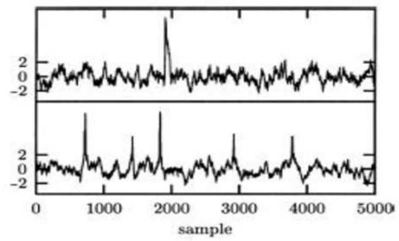
testing for nonlinearity **what can go wrong?**

field applications

- all issues related to embedding
- all issues related to estimating a given dynamical invariant

- issues related to Fourier transform
  finite data, end-point-mismatch (shifting possible?, shortening?)
- non-stationarity vs. nonlinearity
  would need to include (all forms! of) non-stationarity in null
- how to treat singular (extreme) events?
- avoid wishful thinking!
  (sometimes it's just P2C2E*)

T. Schreiber, A. Schmitz / Physica D 142 (2000) 346–382



*process too complicated to explain: Iff in S. Rushdie: Haroun and the Sea of Stories. Granta Books, London (1990)