Measuring Interactions

from Time Series

phase-based techniques

measuring interactions phase

basic idea: interaction ⇔ phase dynamics ⇔ synchronization

- the “stronger” the interaction, the more similar are the phases
- interaction-induced perturbation of phases indexes directionality

neglect amplitude information

phase ⇔ nonlinearity → interesting for chaotic systems

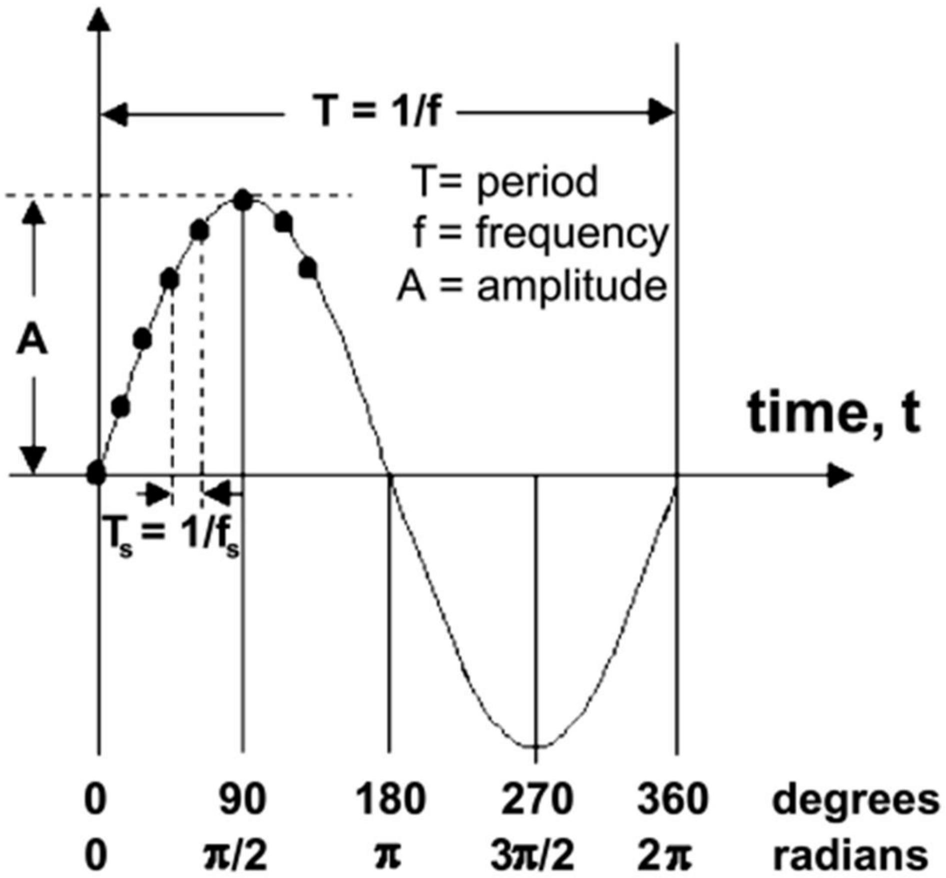
need to derive phase from time series

measuring interactions phase

the phase of a periodic function of some real variable t is the relative value of that variable within the span of each full period

phase is typically expressed as an angle $\phi(t)$

$\phi(t) \in [0°, 360°)$ or $\phi(t) \in [0, 2\pi)$



from: en.wikipedia.org

measuring interactions phase

phase synchronization

classical definition ("phase locking")

"adjustment of rhythms of oscillating objects due to a weak interaction"

$$n\phi_1(t) - m\phi_2(t) = \mathrm{const}; \quad (n,m) \in \mathbb{N}$$

extension for noisy <u>and</u> chaotic systems ("weak phase locking"):

"induction of relationships between the functionals of two processes due to an interaction"

$$n\phi_1(t) - m\phi_2(t) < \mathrm{const}; \quad (n,m) \in \mathbb{N}$$

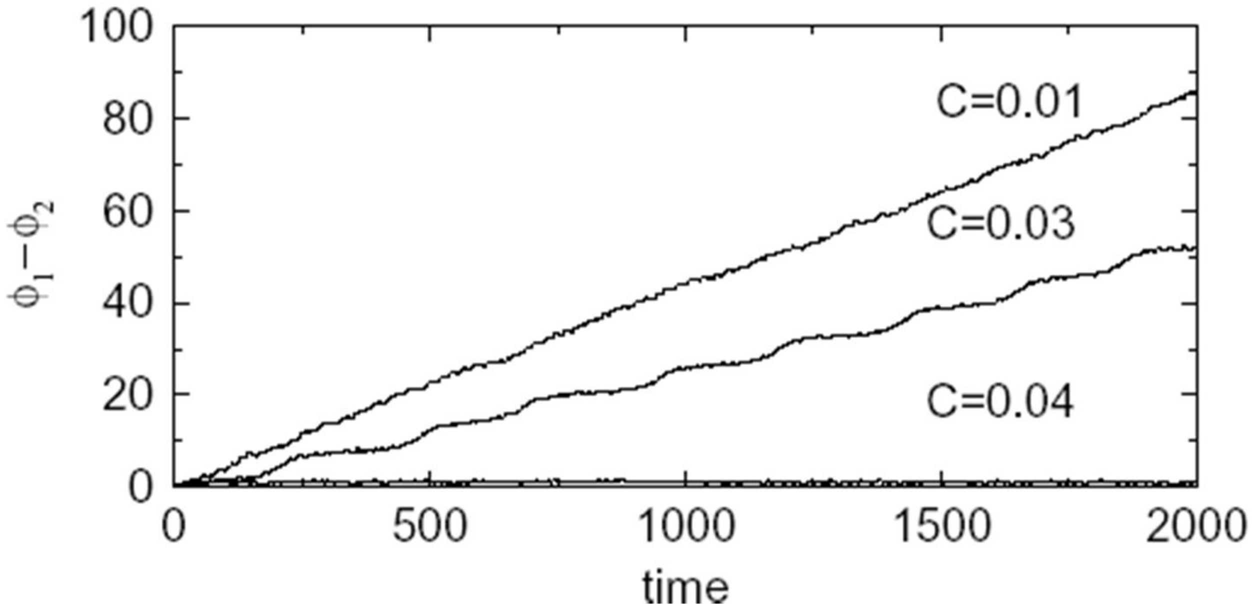
M. Rosenblum et al., Phase synchronization of chaotic oscillators. Phys. Rev. Lett. 76, 1804, 1996

measuring interactions phase

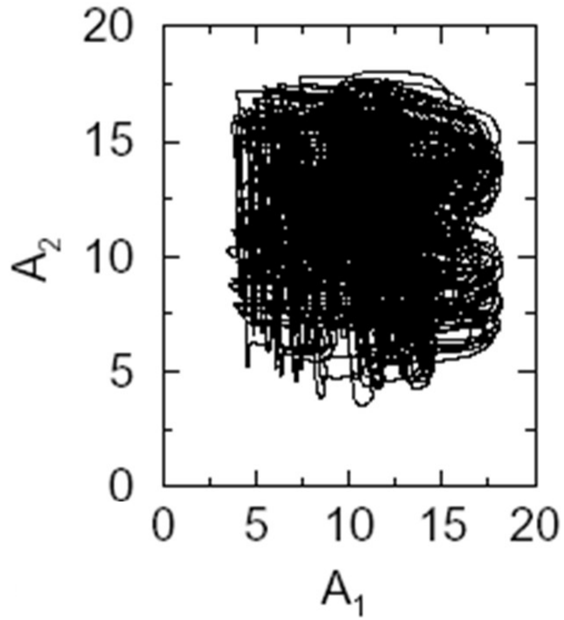
phase synchronization of chaotic oscillators

example: non-identical uni-directionally coupled Rössler oscillators
(C = coupling strength)

adjustment of phases
(from phase-slips to phase-locking)



amplitudes remain
uncorrelated and chaotic

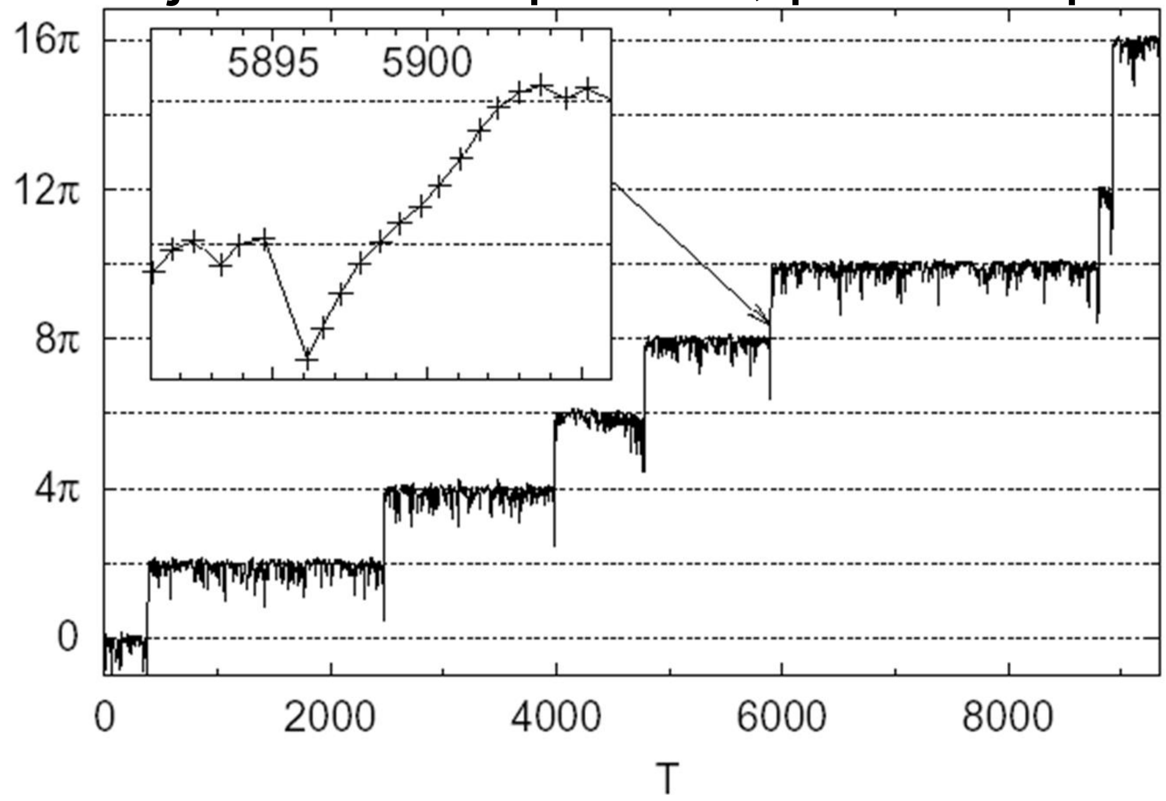


measuring interactions phase

phase synchronization of chaotic oscillators

example: non-identical uni-directionally coupled Rössler oscillators (C = coupling strength)

adjustment of phases; phase-slips



devil's staircase

measuring interactions **deriving phases from time series**

given: time series $\boldsymbol{v}$ of some observable $\boldsymbol{x}$ and

time series $\boldsymbol{w}$ of some observable $\boldsymbol{y}$

and

given some reference time point (e.g. $t_0 = 0$)

(holds for strictly periodic functions, triggered data, impulse responses, transfer functions)

→ define phase relative to t_0

let $v(t) = A(t)\cos(\phi(t_0 + t))$ (analogously for $w(t)$)

with $z(t) = A(t)\exp(i\phi(t_0 + t))$, we have:

$$
\begin{aligned}
v(t) &= \Re(z(t)) \\
\phi(t) &= \arctan\left(\frac{\Re(z(t))}{\Im(z(t))}\right) \\
A(t) &= \sqrt{(\Re(z(t))^2 + (\Im(z(t))^2}
\end{aligned}
$$

derive imaginary part
with e.g. Fourier transform

measuring interactions **deriving phases from time series**

given: time series $\boldsymbol{v}$ of some observable $\boldsymbol{x}$ and
time series $\boldsymbol{w}$ of some observable $\boldsymbol{y}$

if no reference time point given
(arbitrary real-valued signals)

→ define phase using

- zero-crossings (or other marker-events; Rice, 1944)
- Hilbert-transform (Gabor, 1946; Panter 1965)

(- wavelet-transform (Lachaux et al., 1999)*)

*A. Bruns. Fourier-, Hilbert-and wavelet-based signal analysis: are they really different approaches? *J. Neurosci. Methods*, *137*, 321, 2004.

measuring interactions **deriving phases from time series**

phases from zero-crossings

ansatz: successive zero-crossings correspond to completion of a period

- subtract mean value (*demeaning*) if necessary

- let t_k denote beginning of a period and t_{k+1} the next (e.g. via: $v^- \rightarrow 0$ or $v^+ \rightarrow 0$)

- derive phase with linear interpolation:

$$\phi(t) := \frac{t-t_k}{t_{k+1}-t_k} 2\pi + k2\pi; \quad \text{where } t_k \leq t < t_{k+1}$$

measuring interactions **deriving phases from time series**

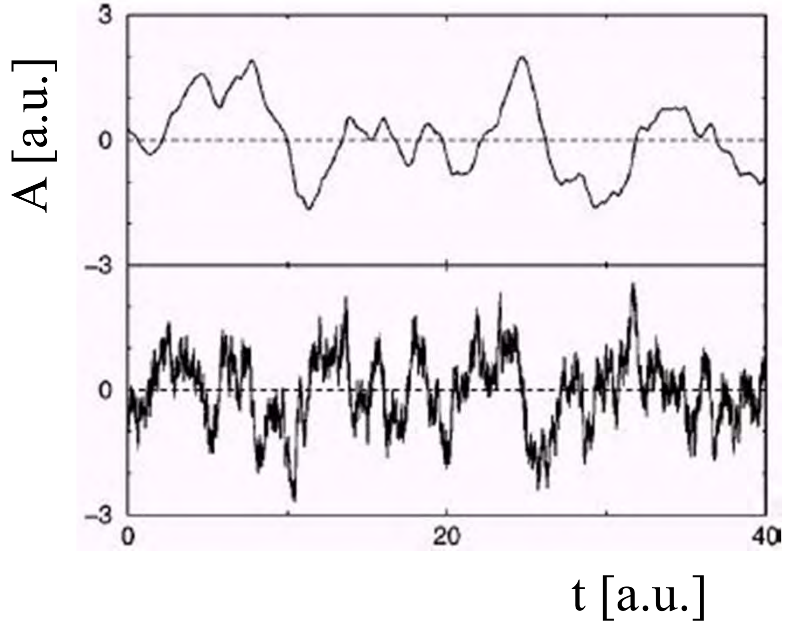
phases from zero-crossings

pros:

easy
fast computation

cons:

requires some periodicity
requires smooth signals
extremely sensitive to noise



from: Callenbach et al., PRE 65, 051110, 2002

measuring interactions **deriving phases from time series**

phases from Hilbert-transform

ansatz: define *instantaneous* phase $\phi(t)$ using the analytic signal:

$$z(t) = v(i) + i\tilde{v}(t)$$

Hilbert-transform $\mathcal{HT}$ is defined as:

$$\mathcal{HT}\,(v(t)) = \tilde{v}(t) := v(t) * \frac{1}{\pi t} = \frac{1}{\pi}\mathrm{p.v.}\int_{-\infty}^{+\infty} \frac{v(\tau)}{t-\tau}\mathrm{d}\tau$$

p.v. = Cauchy principal value

measuring interactions deriving phases from time series

phases from Hilbert-transform

representation in frequency domain:
(use Fourier transform $\mathcal{FT}$ and convolution theorem)

$$\text{with } \mathcal{FT}(\tfrac{1}{\pi t}) = -i\,\text{sign}(\omega) = \begin{cases} +i: & \text{if } \omega < 0 \\ 0: & \text{if } \omega = 0 \\ -i: & \text{if } \omega > 0 \end{cases}$$

$$\text{we have: } \tilde{v}(t) = \mathcal{FT}^{-1}\left[\mathcal{FT}(v(t))\,(-i)\,\text{sign}(\omega)\right]$$

and:

$$\phi(t) = \arctan\frac{\tilde{v}(t)}{v(t)}$$ instantaneous phase (possibly phase unfolding required)

$$\omega(t) = \dot{\phi}(t)$$ instantaneous frequency

spectral power remains unchanged; phases of Fourier spectrum shifted by $\pi/2$

measuring interactions deriving phases from time series

phases from Hilbert-transform

basic example: strongly periodic oscillation

$$\text{given } v(t) = A\cos(\omega t); \quad \text{with } (A, \omega) = \text{const}$$

$$\mathcal{HT}\,(v(t)) := \tilde{v}(t) = -A\sin(\omega t)$$

$$\Rightarrow \phi(t) = \arctan\left(-\frac{\sin\omega t}{\cos\omega t}\right) = -\omega t$$

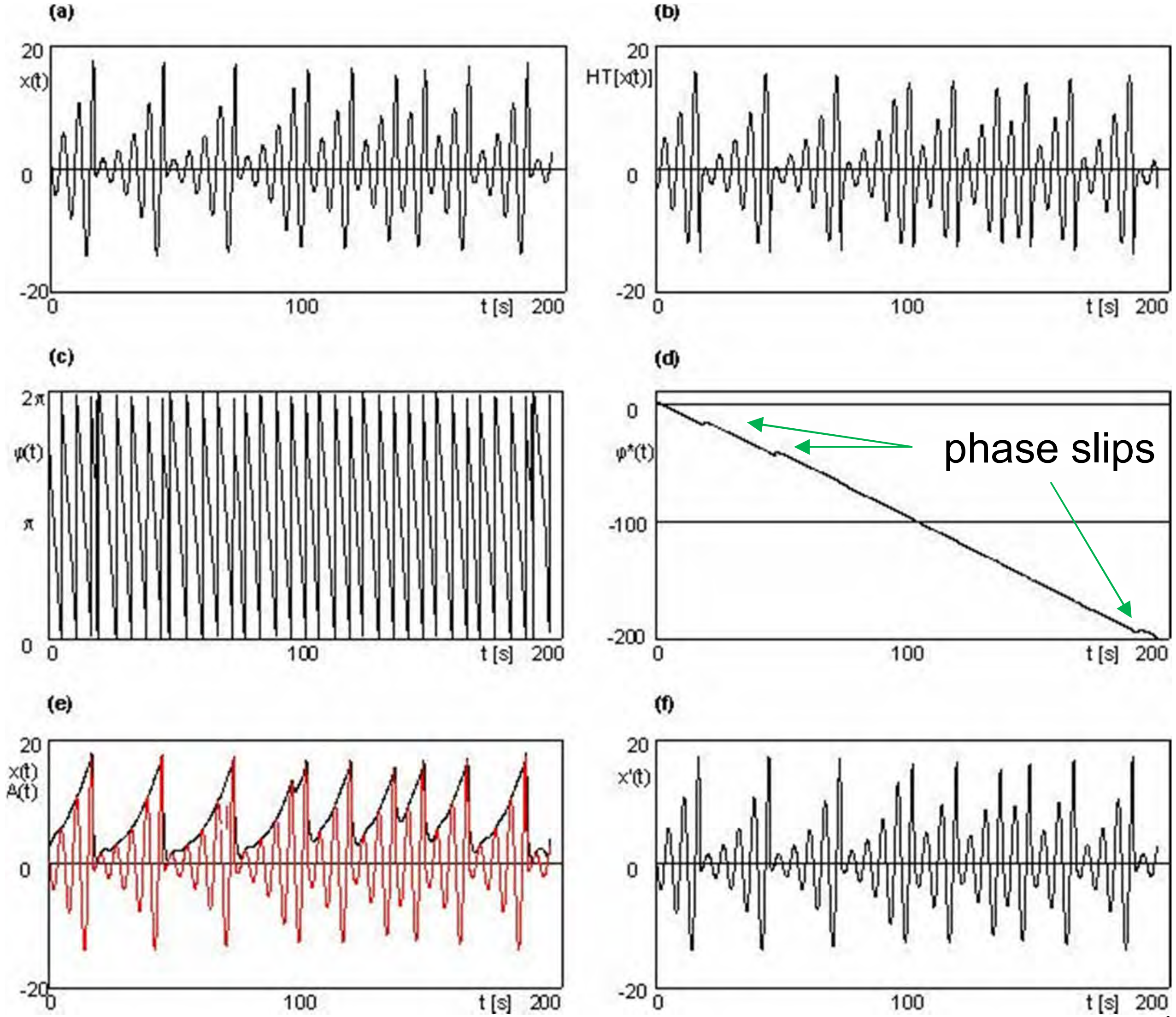
sign: historical reasons

measuring interactions — deriving phases from time series

phases from Hilbert-transform

example: x-component of Rössler oscillator

$$\dot{x} = -0.89y - z$$
$$\dot{y} = x + 0.165y$$
$$\dot{z} = 0.2 + z(x - 10)$$



measuring interactions **deriving phases from time series**

phases from Hilbert-transform

time series analysis: how to

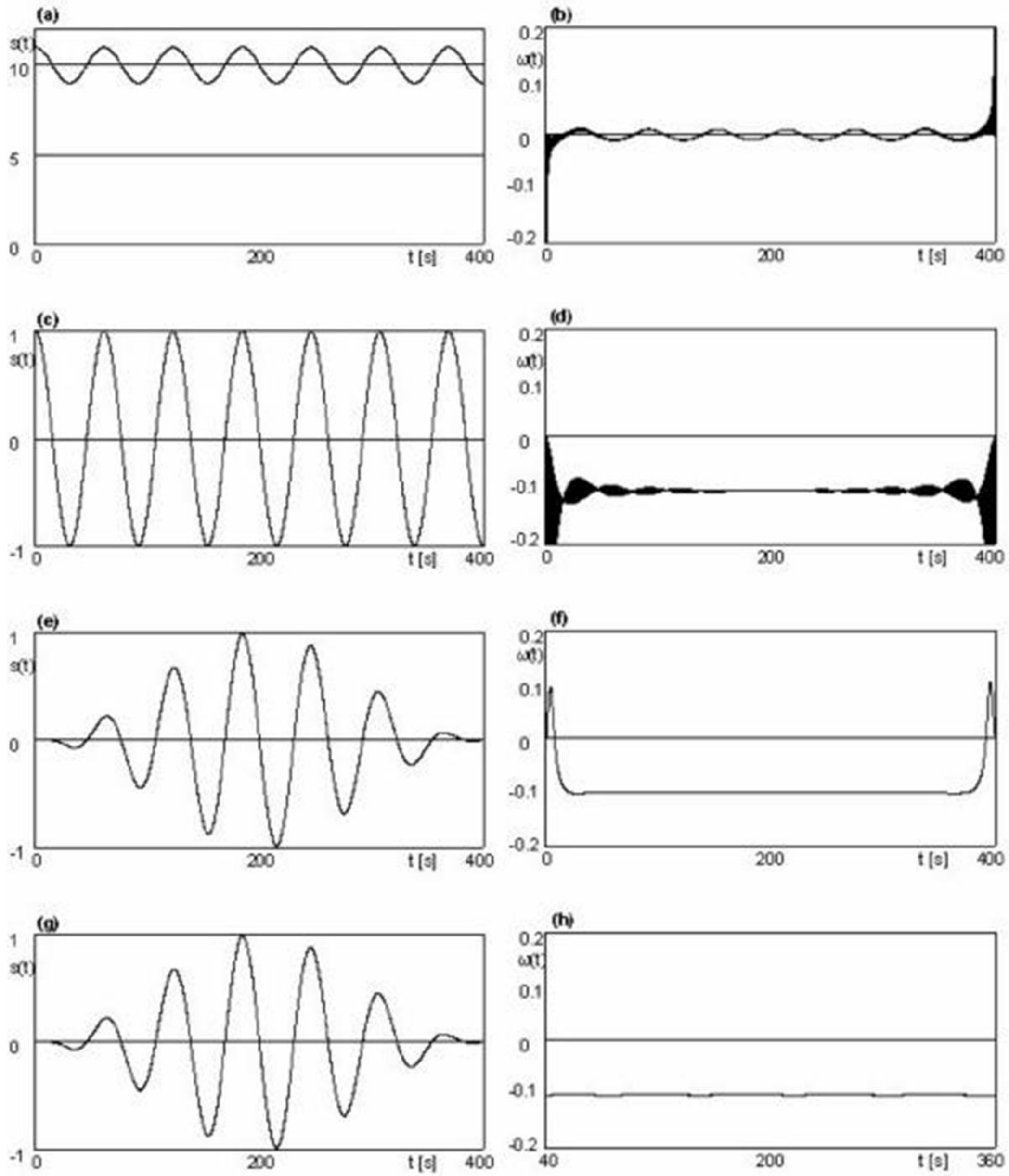
- some periodicity required
- duration of signal: sampling theorem; about 20 data points/period;
  stationarity (at least approximate)
- offsets not taken into account, needs demeaning (subtract mean)
- if you use FFT on finite time series:
  requires trimming/tapering
  tapering can lead to distortions
- unfolding of phases required
- computational speed: $O(N \log(N))$

measuring interactions

deriving phases from time series

phases from Hilbert-transform

time series analysis: how to



measuring interactions strength of interaction

- given phase time series from time series ***v*** and ***w***
- phase synchronization if phase difference time series bounded:

$$\Delta\phi_{\mathrm{vw}}(t) := n\phi_{\mathrm{v}}(t) - m\phi_{\mathrm{w}}(t) \leq \mathrm{const}; \quad \forall t$$

- if phases derived from Hilbert transform: any (n, m)
- if phases derived from zero-crossings: $n = m = 1$
- need to test boundedness

measuring interactions **strength of interaction**

statistical ansatz: ***mean phase coherence****

- phase differences limited to $(0,2\pi]$ (due to arcus tangens)
  → natural circularity → circular statistics**
- estimate moments of "*circular distributions*" by transforming the phase differences onto unit circle in complex plane

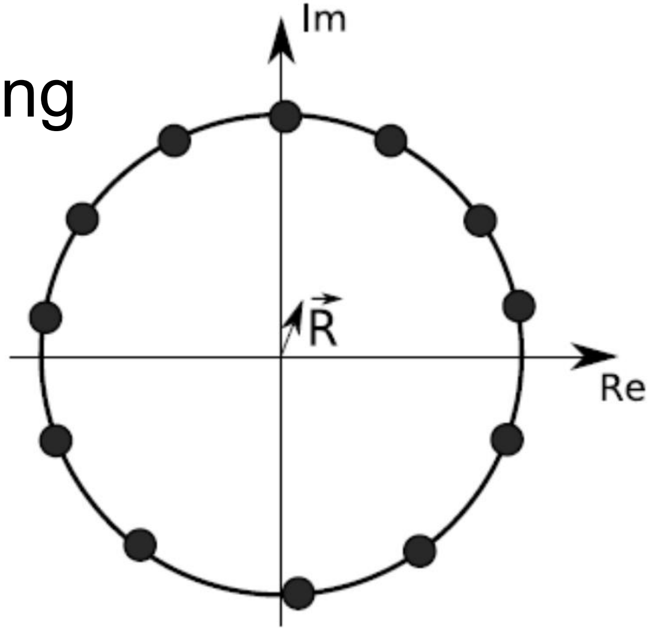
$$\begin{aligned}
\Delta\phi_{\mathrm{vw}}(t) &= n\phi_{\mathrm{v}}(t) - m\phi_{\mathrm{w}}(t) \leq \mathrm{const} \\
z(t) &= \cos\left(\Delta\phi_{\mathrm{vw}}(t)\right) + i\sin\left(\Delta\phi_{\mathrm{vw}}(t)\right) = \exp i(\Delta\phi_{\mathrm{vw}}(t)) \\
Z &= \frac{1}{T}\sum_{t=1}^{T} z(t) \\
R &:= |Z| = \sqrt{(\Re Z)^2 + (\Im Z)^2}
\end{aligned}$$

*M Hoke, K Lehnertz, C Pantev, B Lütkenhöner. Spatiotemporal aspects of synergetic processes in the auditory cortex as revealed by magnetoencephalogram, in: E. Basar, T.H. Bullock (Eds.), Series in Brain Dynamics, Vol. 2, Springer, Berlin, 1989. F Mormann, K Lehnertz, P David, CE Elger. Mean phase coherence as a measure for phase synchronization and its application to the EEG of epilepsy patients. Physica D, 144, 358, 2000
** e.g. KV Mardia, P Jupp. *Directional Statistics*, John Wiley and Sons Ltd., 2000

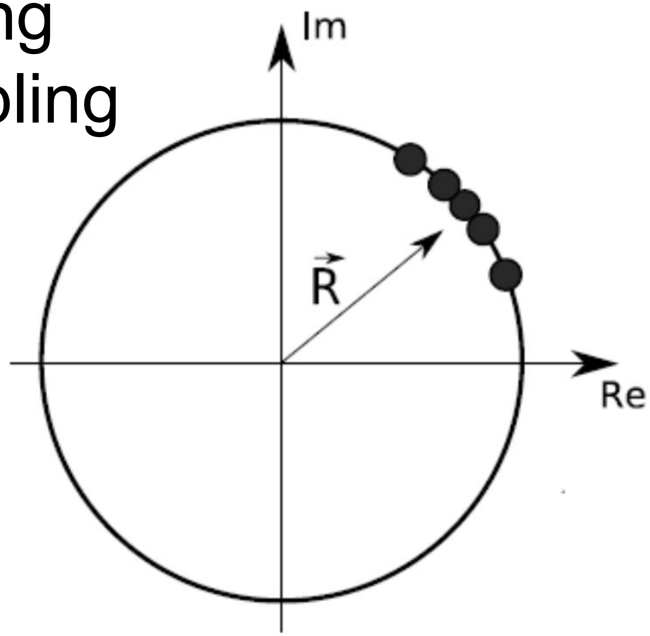
measuring interactions **strength of interaction**

statistical ansatz: ***mean phase coherence***

weak coupling



strong coupling



$R \in [0,1]$

$R = 1$ complete phase synchronization (full phase locking)

$R = 0$ no phase synchronization

$S = 1 - R$ (circular variance)

measuring interactions — strength of interaction

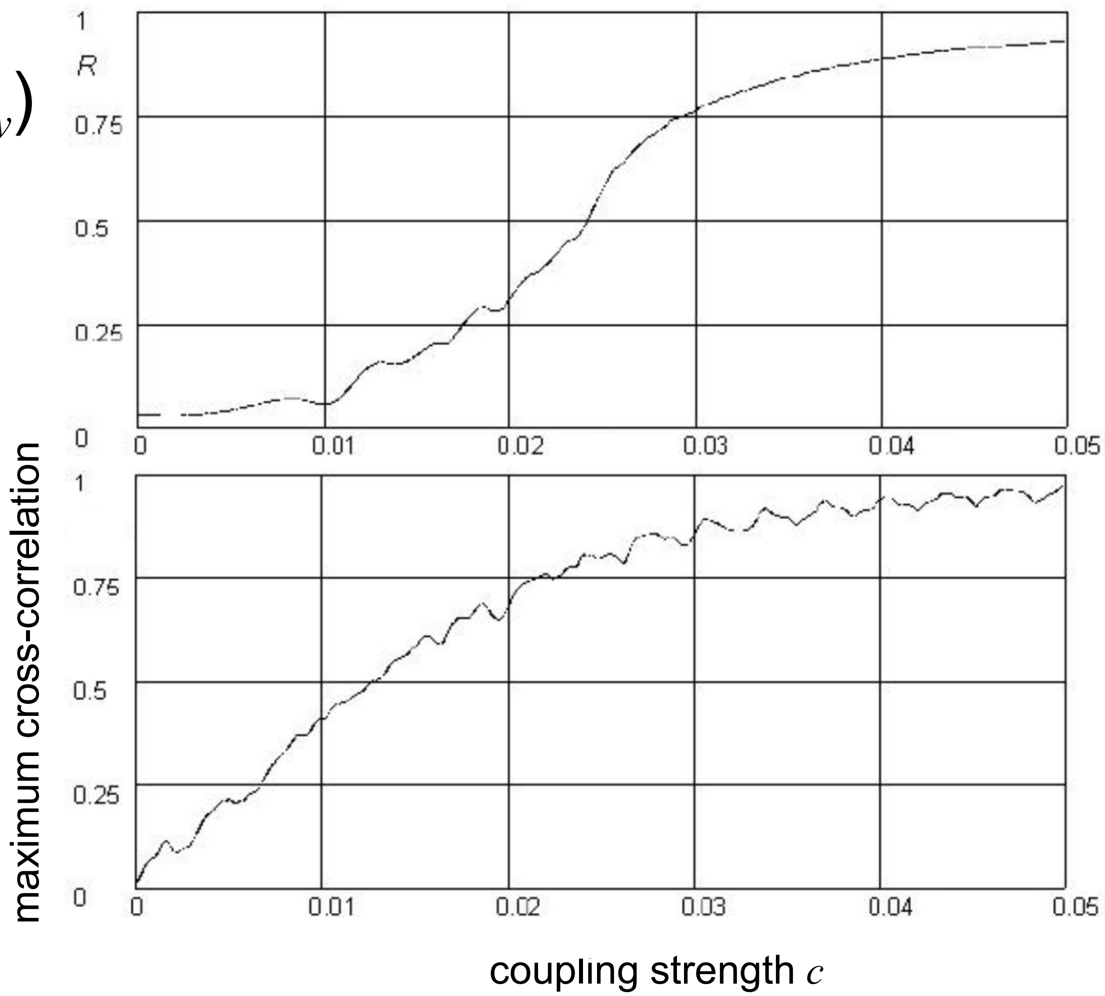
statistical ansatz: ***mean phase coherence*** — ***examples***

diffusively coupled oscillators
(with eigen-frequencies Ω_v and Ω_w)

$$\ddot{\mathbf{v}} = -\Omega_v \mathbf{v} + c(\mathbf{w} - \mathbf{v})$$

$$\ddot{\mathbf{w}} = -\Omega_w \mathbf{w} + c(\mathbf{v} - \mathbf{w})$$

single realization



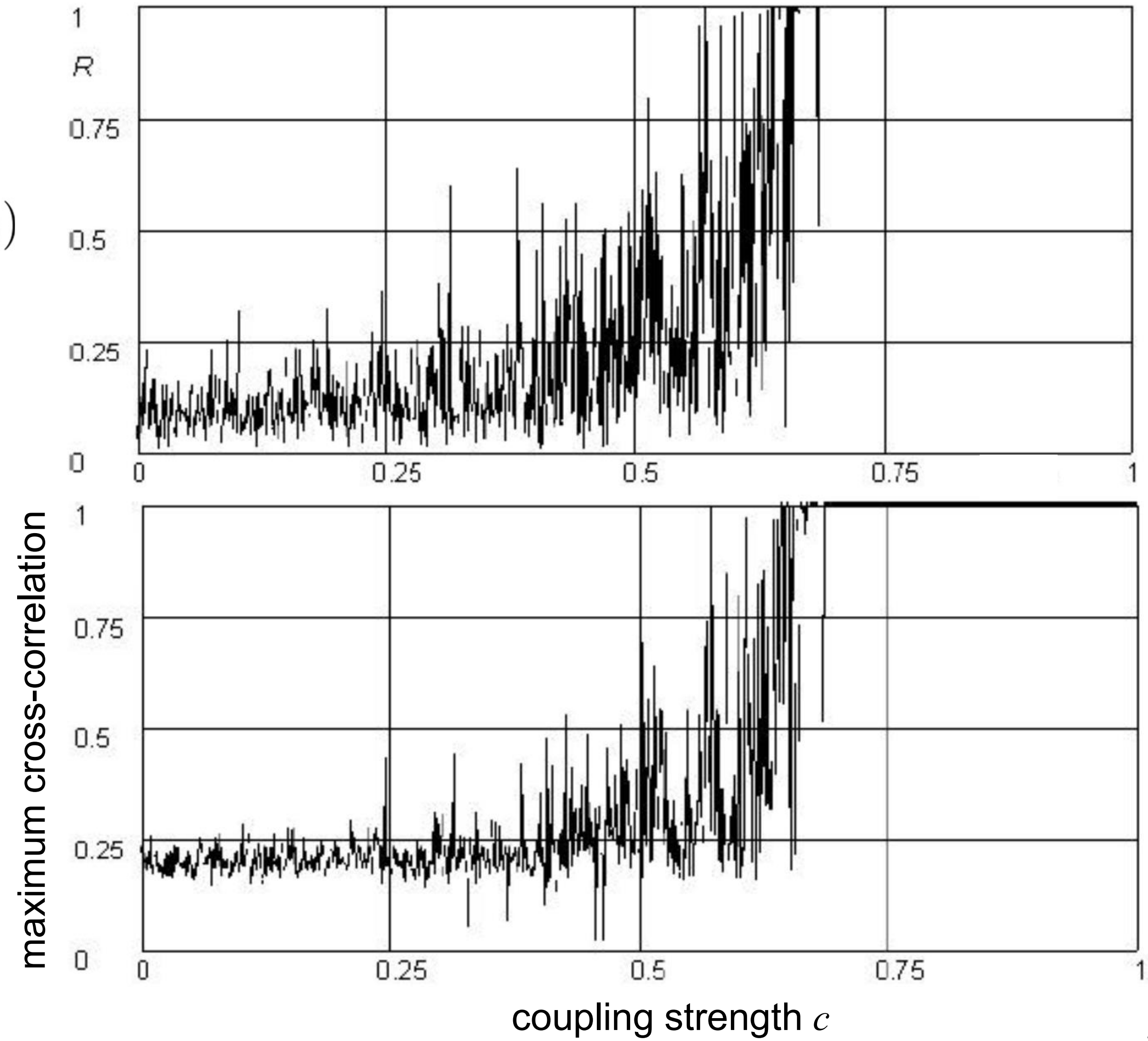
measuring interactions **strength of interaction**

statistical ansatz: ***mean phase coherence*** ***examples***

diffusively coupled
Lorenz oscillators

$$\begin{aligned}\dot{x}_{1,2} &= -\frac{8}{3}x_{1,2} + y_{1,2}z_{1,2} + c(x_{2,1} - x_{1,2}) \\ \dot{y}_{1,2} &= R_{1,2}z_{1,2} - y_{1,2} - x_{1,2}z_{1,2} \\ \dot{z}_{1,2} &= 10(y_{1,2} - z_{1,2}) \\ R_1 &= 28;\ R_2 = 28.00001\end{aligned}$$

single realization



measuring interactions

strength of interaction

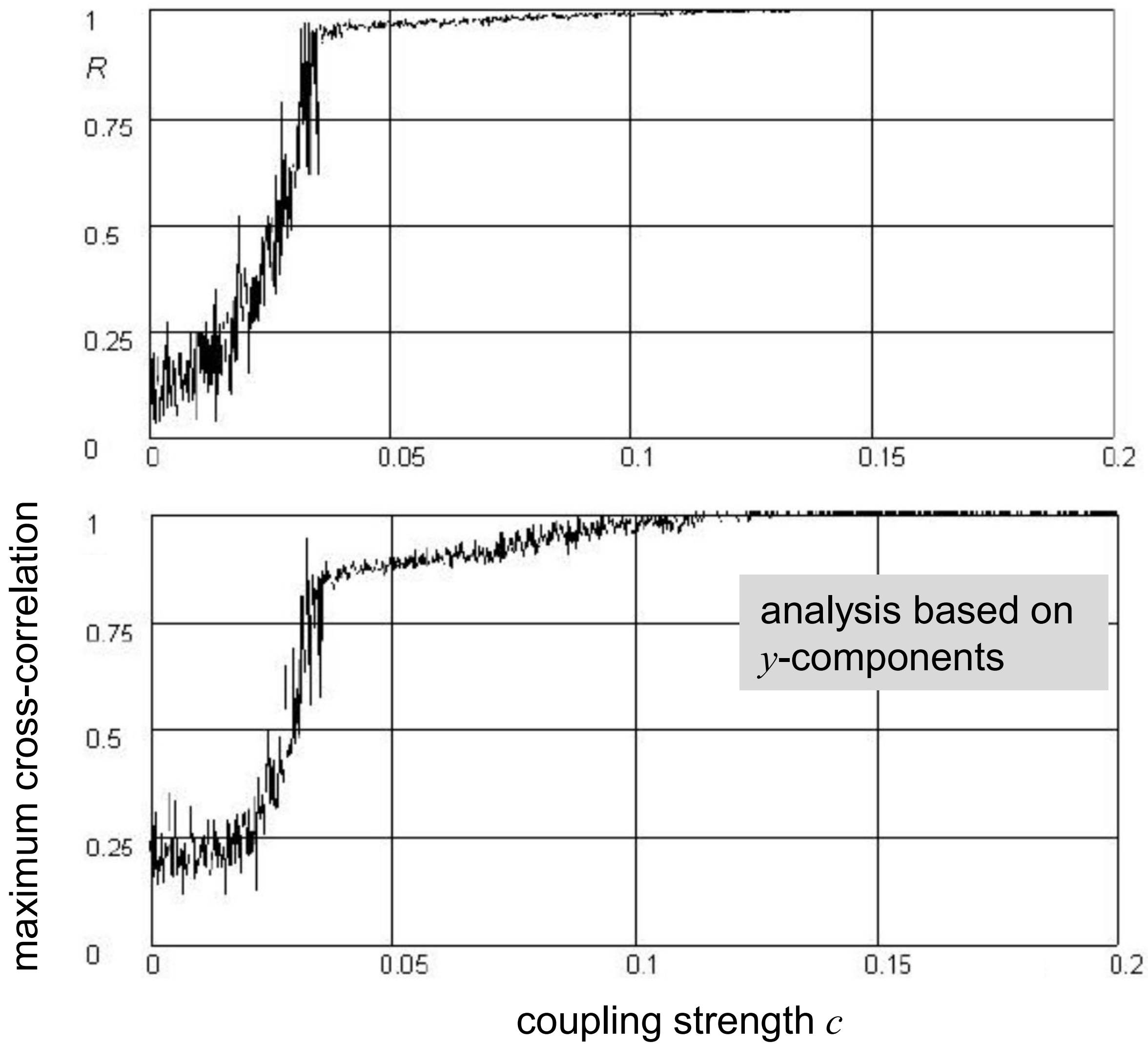
statistical ansatz: ***mean phase coherence***

examples

diffusively coupled Rössler oscillators

$$
\begin{aligned}
\dot{x}_{1,2} &= -\Omega_{1,2}y_{1,2} - z_{1,2} + c(x_{2,1} - x_{1,2}) \\
\dot{y}_{1,2} &= \Omega_{1,2}x_{1,2} + 0.165y_{1,2} \\
\dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10) \\
\Omega_1 &= 0.89;\ \Omega_2 = 0.85
\end{aligned}
$$

single realization



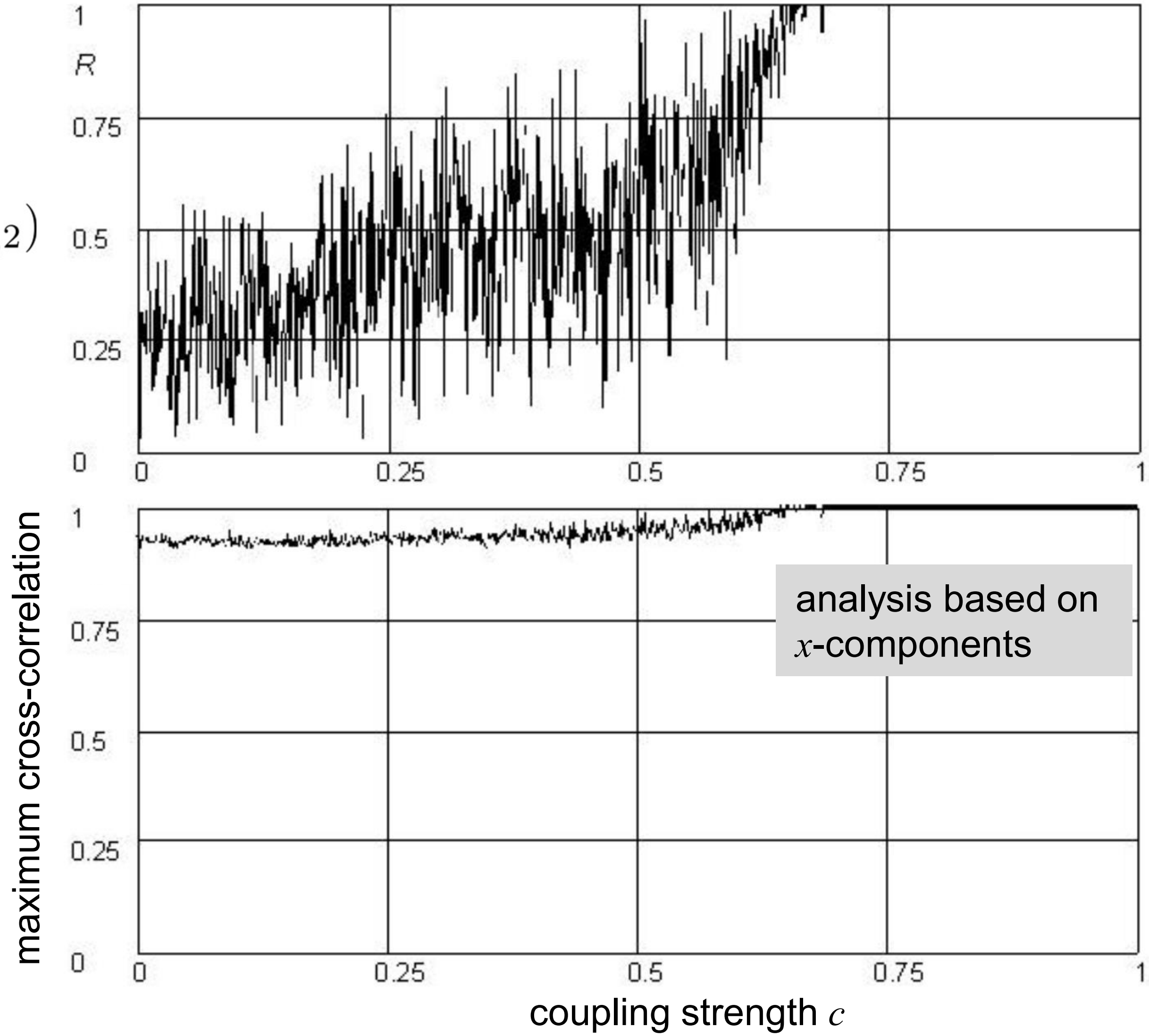
measuring interactions **strength of interaction**

statistical ansatz: ***mean phase coherence*** ***examples***

diffusively coupled
Lorenz oscillators

$$\begin{aligned}\dot{x}_{1,2} &= -\frac{8}{3}x_{1,2} + y_{1,2}z_{1,2} + c(x_{2,1} - x_{1,2}) \\ \dot{y}_{1,2} &= R_{1,2}z_{1,2} - y_{1,2} - x_{1,2}z_{1,2} \\ \dot{z}_{1,2} &= 10(y_{1,2} - z_{1,2}) \\ R_1 &= 28;\ R_2 = 28.00001\end{aligned}$$

single realization



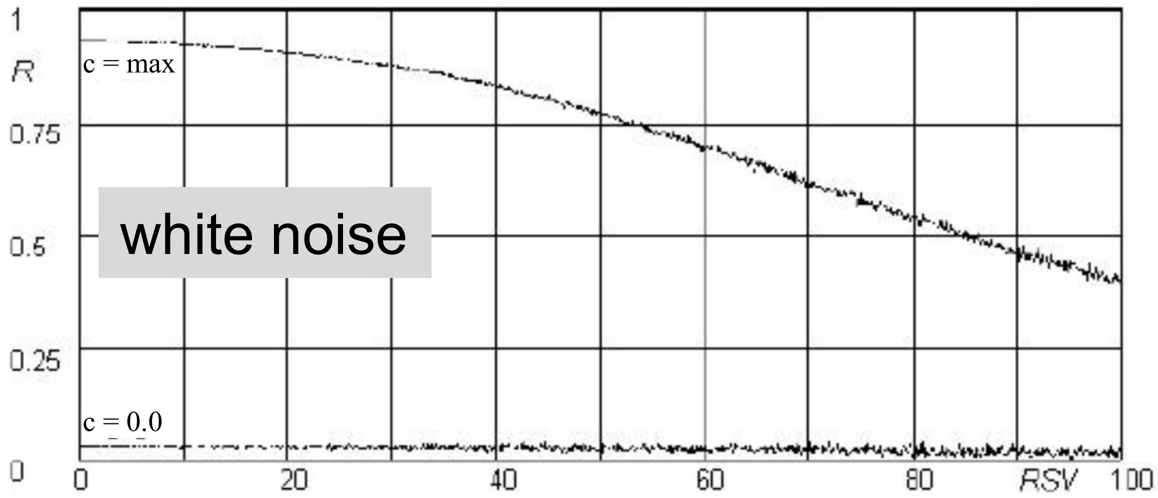
measuring interactions **strength of interaction**

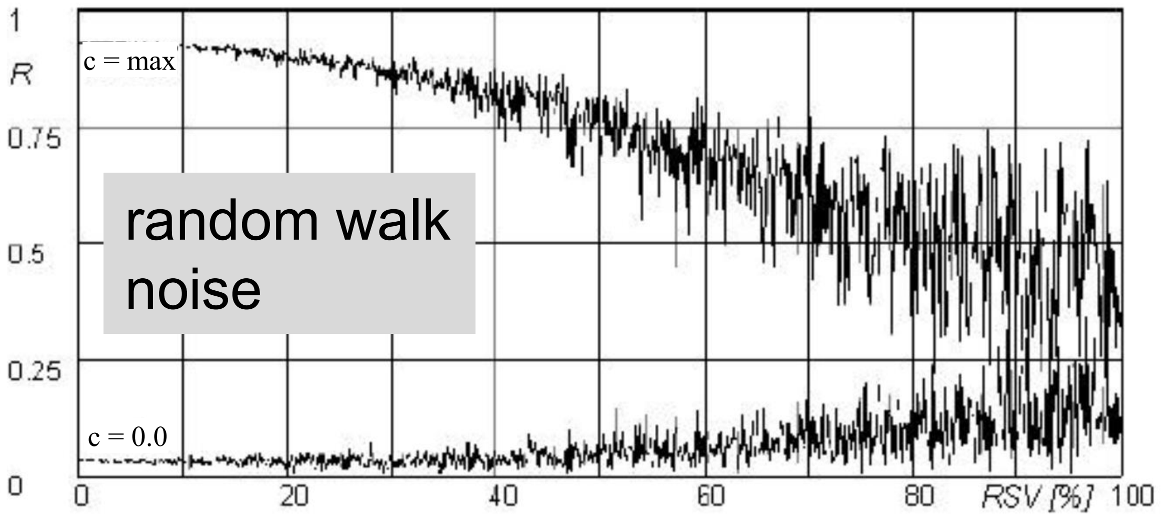
statistical ansatz: ***mean phase coherence*** ***examples***

robustness against noise for diffusively coupled oscillators

no coupling vs. maximum coupling

single realization





$$\mathrm{RSV} = \frac{\sigma_{\text{noise}}}{\sigma_{\text{signal}}}$$

measuring interactions **strength of interaction**

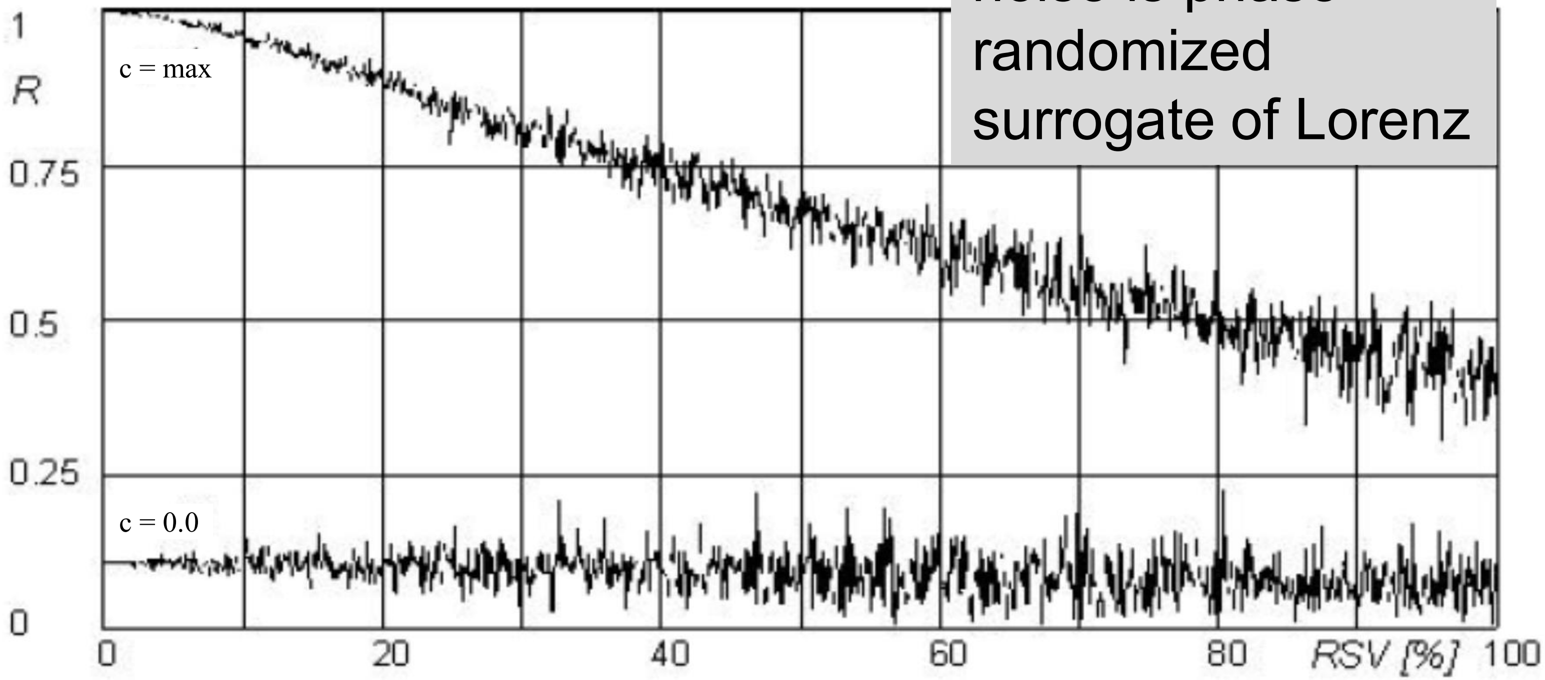
statistical ansatz: ***mean phase coherence*** ***examples***

robustness against noise for diffusively coupled Lorenz oscillators (y-component)

no coupling vs. maximum coupling

single realization

noise is phase-randomized surrogate of Lorenz



$$\mathrm{RSV} = \frac{\sigma_{\mathrm{noise}}}{\sigma_{\mathrm{signal}}}$$

measuring interactions **strength of interaction**

- other approaches (based on information theory; Tass et al., 1998)
  - index based on Shannon entropy
  - index based on conditional probability

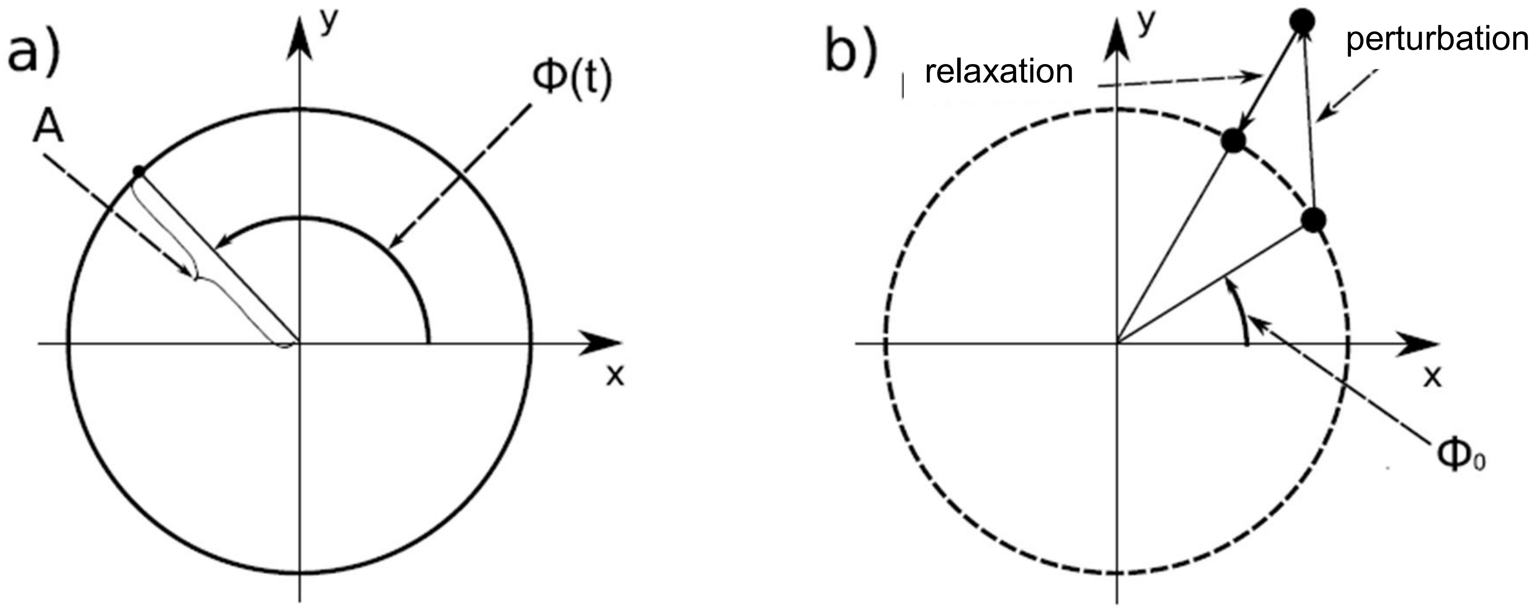
- different approaches yield similar findings (application dependent)

- mean phase coherence most robust, easiest to estimate, wide applicability

- mean phase coherence and related approaches are symmetric (under exchange of v and w)
  - → can not indicate direction of interaction

measuring interactions **direction of interaction**

- given phase time series from time series v and w
- observation: weak interaction induces perturbation of phases dynamics (perturbation of amplitudes can be neglected)



- need a characterization of mutual perturbations of phase dynamics

measuring interactions — direction of interaction

from simplified phase model to ***cross dependencies***
(evolution map approach*)
- assumption: weakly coupled, self-sustained oscillators

$$\dot{\phi}_1(t) = \omega_1 + \kappa_1 f_1(\phi_1(t), \phi_2(t)) + \xi_1(t)$$
$$\dot{\phi}_2(t) = \omega_2 + \kappa_2 f_2(\phi_2(t), \phi_1(t)) + \xi_2(t)$$

$f_{1,2}(\phi_{1,2})$ are coupling functions
$\kappa_{1,2}$ are damping constants
stochastic components $\xi_{1,2}$ for noisy or chaotic oscillators
weak coupling if $\kappa_{1,2} f_{1,2} \ll \omega_{1,2}$

* MG Rosenblum, AS Pikovsky. Detecting direction of coupling in interacting oscillators. Phys. Rev. E, 64:045202(R), 2001.

measuring interactions

direction of interaction

from simplified phase model to ***cross dependencies***

- define cross dependencies $d^{(1,2)}$

$$c_{1,2}^2 := \left\langle \left(\frac{\partial \dot{\phi}_{1,2}}{\partial \phi_{2,1}} \right)^2 \right\rangle = \kappa_{1,2} \left\langle \left(\frac{\partial f_{1,2}}{\partial \phi_{2,1}} \right)^2 \right\rangle$$

$$\text{where } \langle (\cdot) \rangle = \iint_0^{2\pi} (\cdot) \mathrm{d}\phi_{1,2} \mathrm{d}\phi_{2,1}$$

$$d^{(1,2)} := \frac{c_2 - c_1}{c_1 + c_2} \quad \text{with } d^{(1,2)} \in [-1, 1]$$

* MG Rosenblum, AS Pikovsky. Detecting direction of coupling in interacting oscillators. Phys. Rev. E, 64:045202(R), 2001.

measuring interactions **direction of interaction**

from simplified phase model to ***cross dependencies***

- interpretation of cross dependencies $d^{(1,2)}$

$$d^{(1,2)} = \begin{cases} +1: & \text{if system 1 drives system 1 (unidirectional coupling)} \\ 0: & \text{no driving identifiable (symmetric bidir. coupling)} \\ -1: & \text{if system 2 drives system 1 (unidirectional coupling)} \end{cases}$$

* MG Rosenblum, AS Pikovsky. Detecting direction of coupling in interacting oscillators. Phys. Rev. E, 64:045202(R), 2001.

measuring interactions **direction of interaction**

from simplified phase model to ***cross dependencies***

- numerical estimation of cross dependencies $d^{(1,2)}$

define incremental phase time series

$$\Delta_{1,2}(k) = \phi_{1,2}(t_k + \tau) - \phi_{1,2}(t_k)$$

with the (noisy) mapping

$$\Delta_{1,2}(k) = \mathcal{F}_{1,2}(\phi_{1,2}(t_k), \phi_{2,1}(t_k)) + \xi(t_k)$$

approximate $\mathcal{F}$ with Fourier series using least-squares fit

$$\begin{aligned} \Delta_{1,2}(k) &\overset{\text{min!}}{\approx} F_{1,2}(\phi_{1,2}(t_k), \phi_{2,1}(t_k)) \\ &= \sum_{m,n} a^{1,2}_{m,n} \cos(m\phi_{1,2} + n\phi_{2,1}) \\ &+ b^{1,2}_{m,n} \sin(m\phi_{1,2} + n\phi_{2,1}) \end{aligned}$$

measuring interactions direction of interaction

from simplified phase model to ***cross dependencies***

- numerical estimation of cross dependencies $d^{(1,2)}$

with appropriately chosen orders of Fourier series, one finds:

$$c_{1,2}^2 = \left\langle \left(\frac{\partial \dot{\phi}_{1,2}}{\partial \phi_{2,1}} \right)^2 \right\rangle = 2\pi^2 \sum_{m,n} n^2 \left((a_{m,n}^{1,2})^2 + (b_{m,n}^{1,2})^2 \right)$$

note that the least-squares fit moderately reduces noise

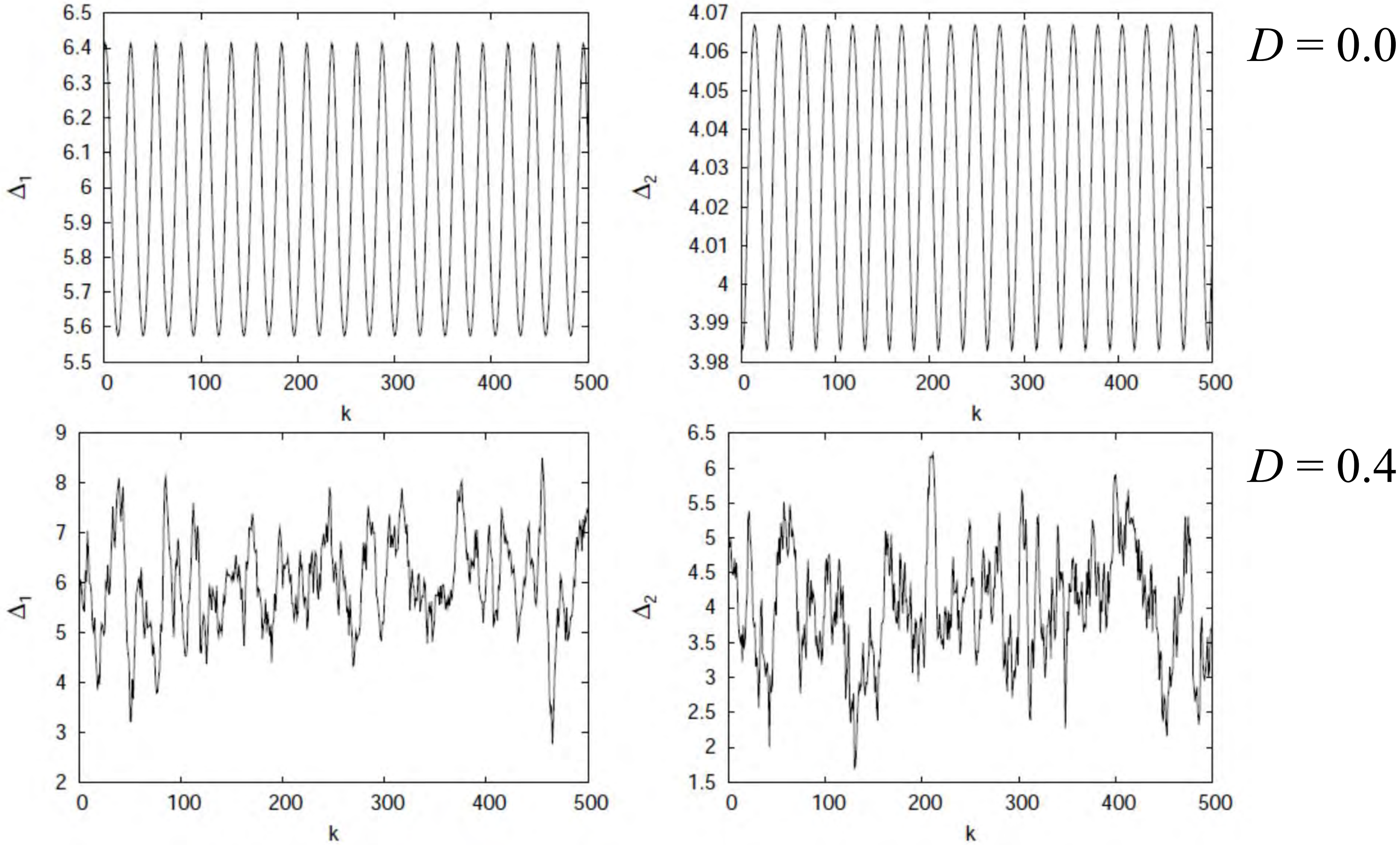
* MG Rosenblum, AS Pikovsky. Detecting direction of coupling in interacting oscillators. Phys. Rev. E, 64:045202(R), 2001.

measuring interactions — direction of interaction

cross dependencies — *examples*

$$\dot{\phi}_{1,2} = \omega_{1,2} + \kappa_{1,2}\sin(\phi_{1,2} - \phi_{2,1}) + D\xi_{1,2}$$

incremental
phase
time series



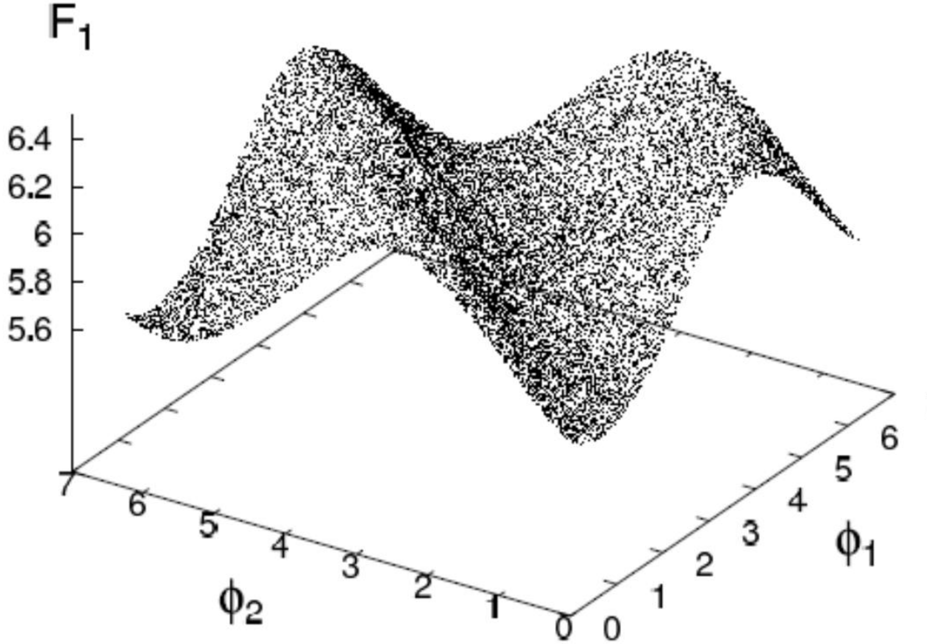
measuring interactions **direction of interaction**

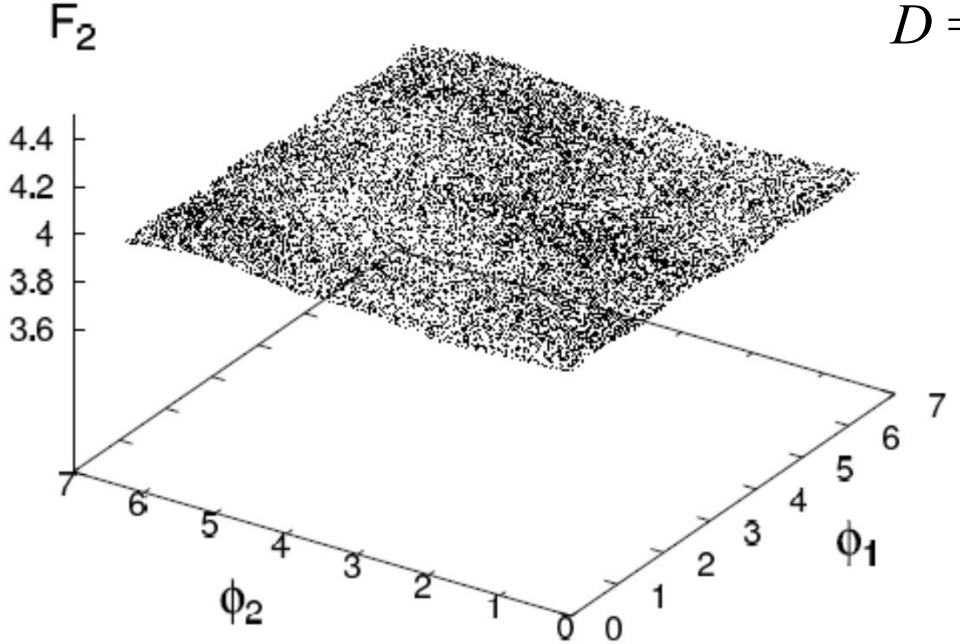
cross dependencies ***examples***

$$\dot{\phi}_{1,2} = \omega_{1,2} + \kappa_{1,2} \sin(\phi_{1,2} - \phi_{2,1}) + D\xi_{1,2}$$

Fourier series

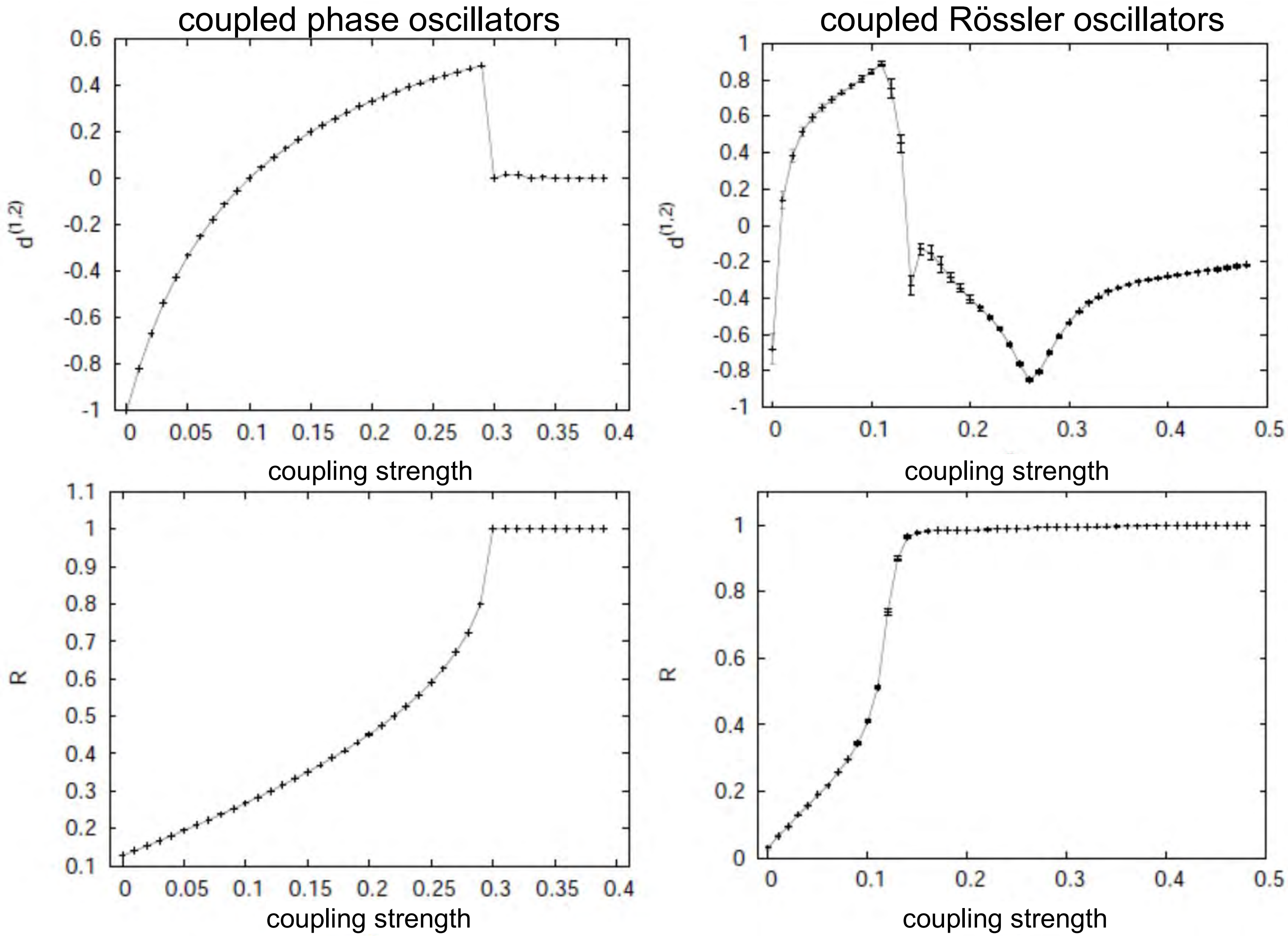
$D = 0.4$





measuring interactions — direction of interaction

cross dependencies — *examples*



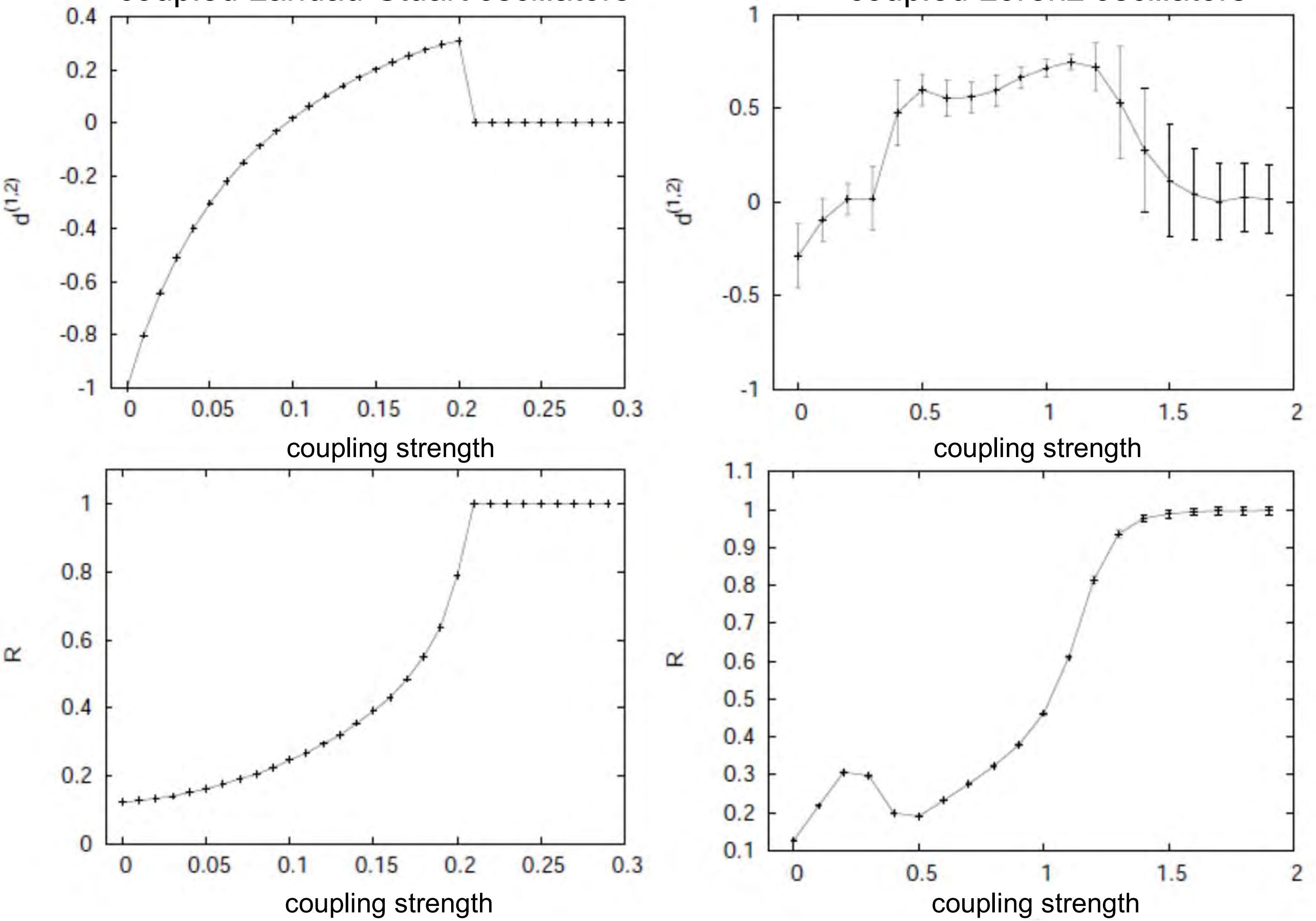
cross dependencies

mean phase coherence

measuring interactions — direction of interaction

cross dependencies — *examples*

coupled Landau-Stuart oscillators — coupled Lorenz oscillators



cross dependencies

mean phase coherence

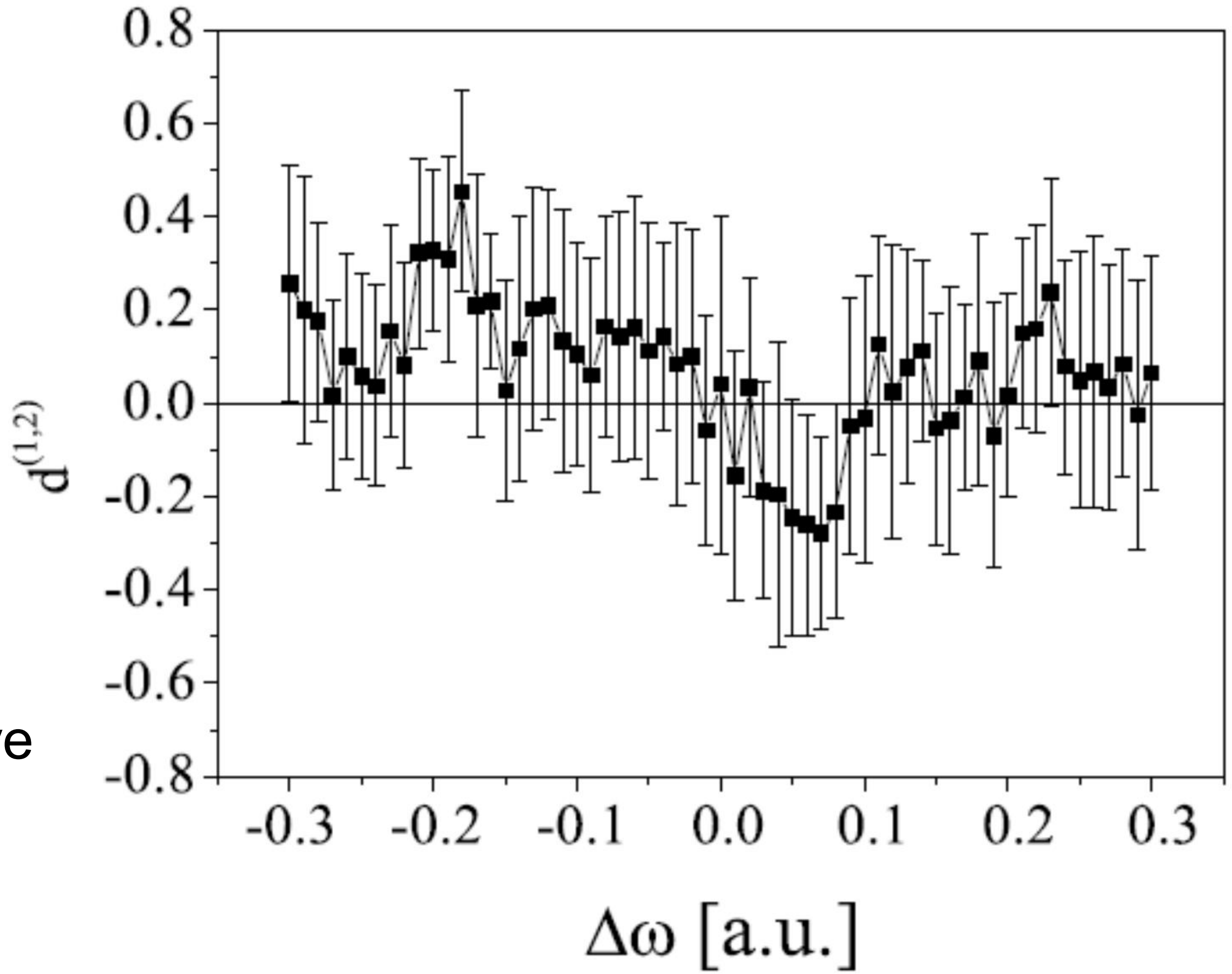
measuring interactions — direction of interaction

cross dependencies — *examples*

uncoupled(!)
Rössler oscillators
with different
eigen-frequencies
$\omega_1 = 0.9; \omega_2 \in [0.6; 1.2]$
(20 realizations)

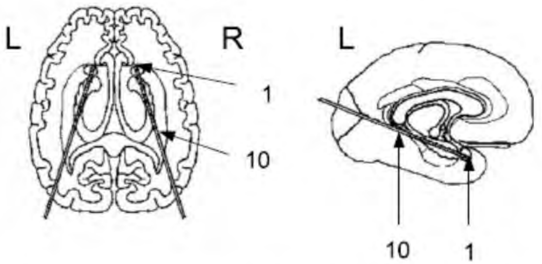
dependence on
frequency detuning $\Delta\omega = \omega_1 - \omega_2$

the fast system appears to drive
the other system

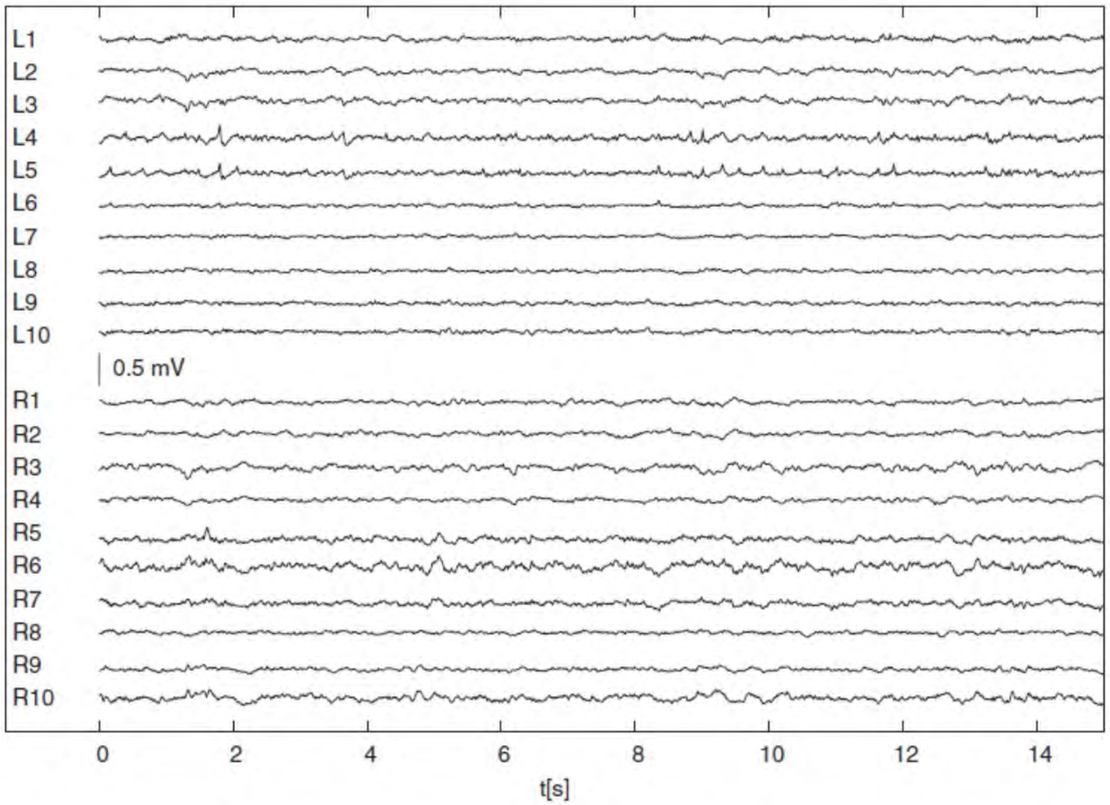


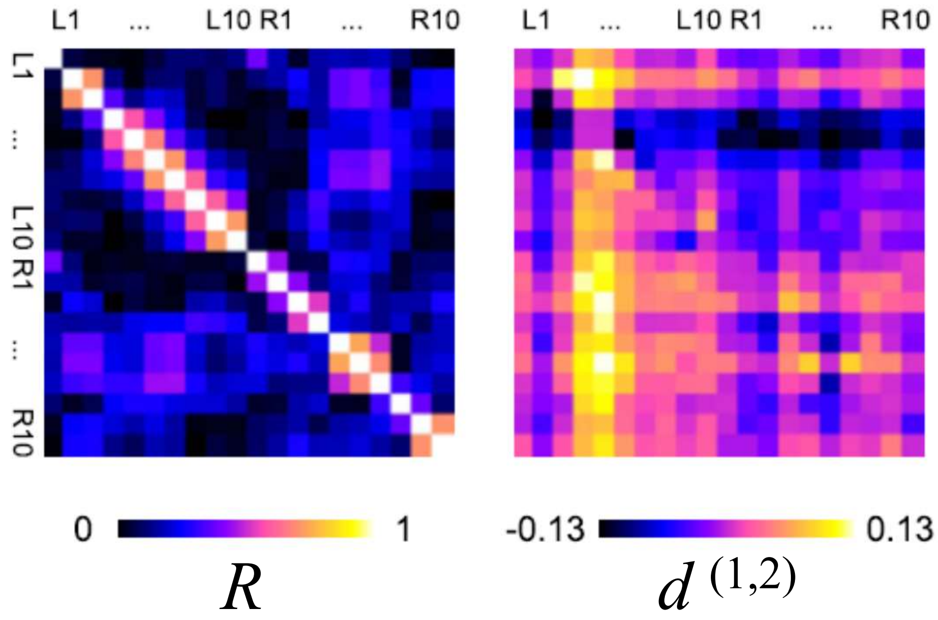
measuring interactions **phase-synchronization**

strength and direction of interaction ***examples***



focal driving in epilepsy





(based on ~ 150 min EEG from seizure-free interval)

Osterhage H, Mormann F, Wagner T, Lehnertz K. Detecting directional coupling in the human epileptic brain: Limitations and potential pitfalls. Phys. Rev E. 77, 011914, 2008

measuring interactions phase-synchronization

strength and direction of interaction

phase-based estimators

extensions

- various data-driven estimation techniques*

- estimators for transient signals**

- multivariate (partial) estimators***

*see, e.g. Porz S, Kiel M, Lehnertz K. Can spurious indications for phase synchronization due to superimposed signals be avoided?. Chaos 24, 033112, 2014.
**Wagner T, Fell J, Lehnertz K. The detection of transient directional couplings based on phase synchronization. New J. Physics 12, 053031, 2010
*** e.g. Schelter B, Winterhalder M, Dahlhaus R, Kurths J, Timmer J. Partial phase synchronization for multivariate synchronizing systems. Phys. Rev. Lett. 96, 208103, 2006;
Kralemann B, Pikovsky A, Rosenblum M. Reconstructing effective phase connectivity of oscillator networks from observations. New J Physics 16, 085013, 2014;
Rings T, Lehnertz K. Distinguishing between direct and indirect directional couplings in large oscillator networks: Partial or non-partial phase analyses?. Chaos 26, 093106, 2016

measuring interactions **phase-synchronization**

strength and direction of interaction

phase-based estimators

advantages

- (relatively) easy-to-use, fast-to-calculate
- high / moderate robustness ($R\ /\ d^{(1,2)}$) against noise

disadvantages

- consider phase information only
- require appropriate choice of algorithmic parameter
- “faster” system (eigen-frequency, noise) → driver (need reliable surrogate test for directionality)
- may be fooled by (unobserved) third system