Measuring Interactions

from Time Series

state-space*-based techniques

*same as phase-space but used here to avoid confusion with “phase”

measuring interactions **state-space**

basic idea: interaction ⇔ dynamics on attractor ⇔ synchronization

- the "stronger" the interaction, the more similar are the attractors
- testing for conditional changes of attractor properties indexes directionality

more comprehensive characterization of interacting nonlinear systems

need to extend the classical concept of synchronization

measuring interactions **state-space**

synchronization

- requires some (self-sustained) oscillatory behavior of autonomous systems

- does not necessarily indicate synchronous motion

- requires weak coupling
  strong coupling → identical motion → not of interest
  (natural phenomena ⇔ weak coupling!)

- is a dynamical phenomenon, not a state !

measuring interactions state-space

given observables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of systems X and Y

complete (or identical) synchronization

$$\lim_{t\to\infty} |\mathbf{x}(t) - \mathbf{y}(t)| \equiv 0$$

lag synchronization

$$\mathbf{x}(t+\tau) = \mathbf{y}(t) \quad \forall t$$

phase synchronization

$$\Delta\phi_{\mathrm{XY}}(t) := n\phi_{\mathrm{X}}(t) - m\phi_{\mathrm{Y}}(t) \leq \mathrm{const}; \quad \forall t \text{ and } (n,m) \in \mathbb{N}$$

measuring interactions state-space

complete synchronization:

identical systems or infinitely strong coupling

lag synchronization:

for $\tau \to 0$: cross-over to complete synchronization

phase synchronization:

intuitively: phase sync. → lag sync. → complete sync.

but: too many counterexamples

these concepts do not capture interactions between attractors

→ need another concept

measuring interactions **state-space**

given observables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of systems X and Y

generalized synchronization (GS)

there is a functional ψ such that (after a transitory evolution from appropriately chose initial condition), we have:

$$\mathbf{x}(t) = \Psi(\mathbf{y}(t))$$

the dynamical state of one of the systems is completely determined by the state of the other

trajectory of joint system confined to sub-manifold of joint state-space

$$\dim(\mathrm{X} \oplus \mathrm{Y}) \leq \dim(\mathrm{X}) + \dim(\mathrm{Y})$$

Fujisaka H, Yamada T. Stability theory of synchronized motion in coupled-oscillator systems. Prog. Theor. Phys 69, 32, 1983
Afraimovich VS, Verichev NN, Rabinovich MI. Stochastic synchronization of oscillation in dissipative systems. Radiophys. Quant. Electr. 29, 795, 1986
Rulkov NF, Sushchik MM, Tsimring LS, Abarbanel HD. Generalized synchronization of chaos in directionally coupled chaotic systems. Phys. Rev E 51, 980, 1995

measuring interactions state-space

generalized synchronization

- if systems are mutually coupled ψ has to be invertible
- for a driver-responder configuration ψ does not need to be invertible
- if ψ is the identity, we have identical synchronization

- one can find phase synchronization in case of generalized synchronization

but:

- generalized synchronization is not a necessary condition for phase synchronization

since 2000: attempts to find a unifying definition for synchronization

e.g. Brown R, Kocarev L. A unifying definition of synchronization for dynamical systems. Chaos 10, 344, 2000

measuring interactions state-space

identifying generalized synchronization

- equations of motion are known a priori
- let d and r denote the dimensions of a driver (D) and a responder system (R), resp.
- let $\lambda_i^{(\mathrm{D})}, i = 1, \ldots, d$ denote the Lyapunov spectrum of the driver system, and $\lambda_i^{(\mathrm{R})}, i = 1, \ldots, r$ the spectrum of the responder system
- generalized synchronization, iff $\lambda_i^{(\mathrm{R})} < 0 \ (\forall i)$

Pecora LM, Carroll TL. Synchronization in chaotic systems. Phys. Rev. Lett. 64, 821, 1990

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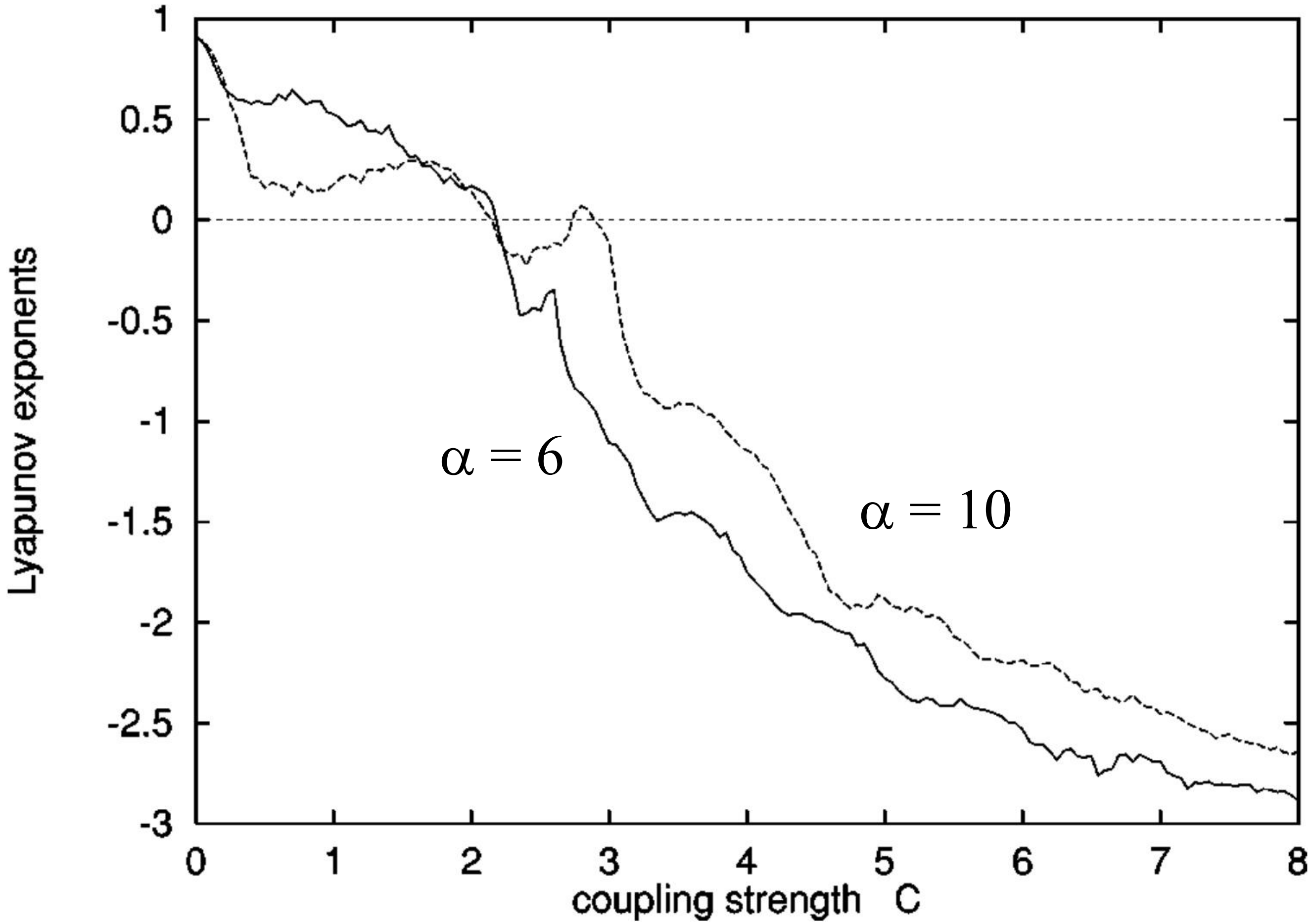
identifying generalized synchronization an example

driver: (modified) Rössler system

$$\dot{x}_1 = -\alpha(x_2 + x_3)$$
$$\dot{x}_2 = \alpha(x_1 + 0.2x_2)$$
$$\dot{x}_3 = \alpha\,(0.2 + x_3(x_1 - 5.7))$$

responder: Lorenz system

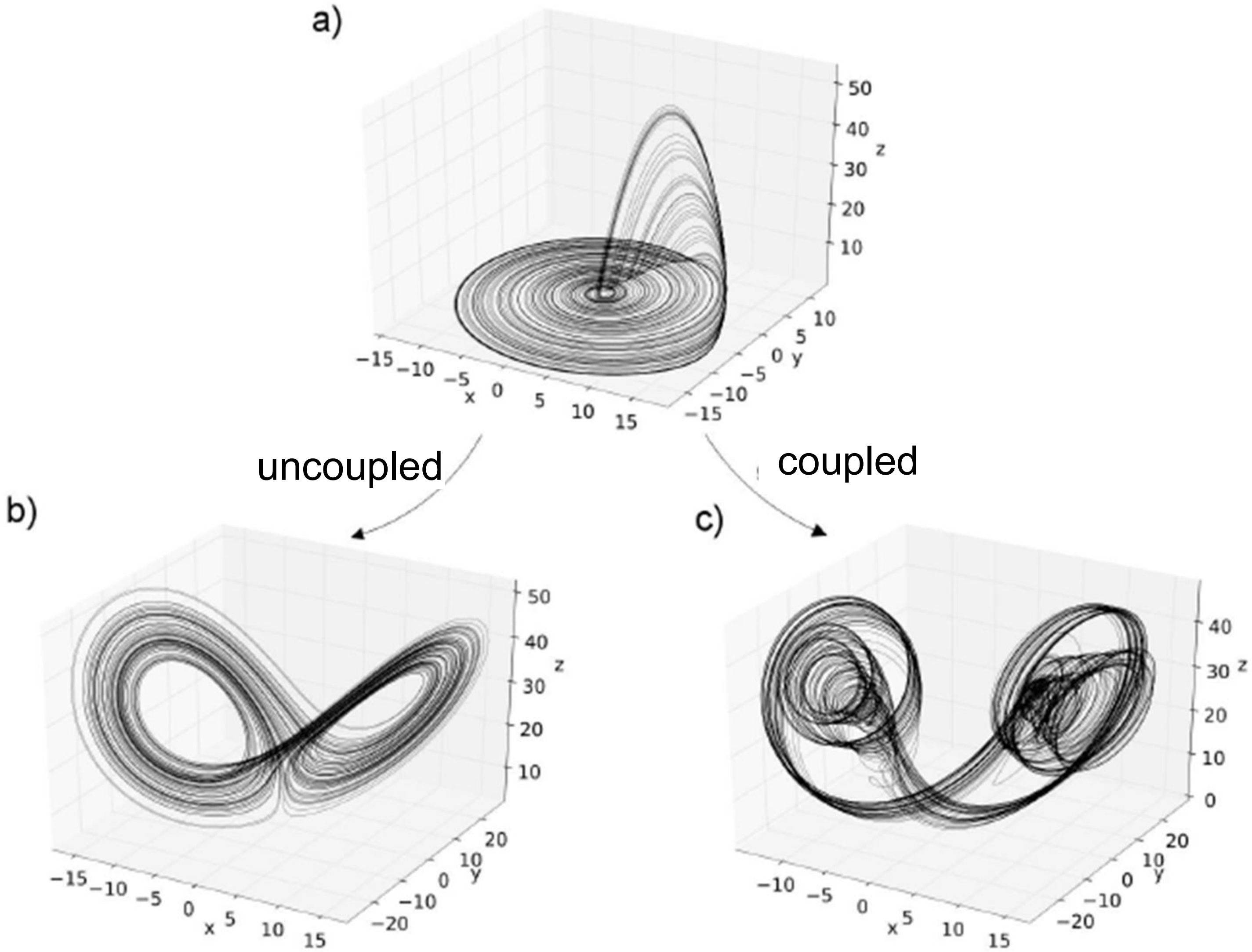
$$\dot{y}_1 = 10(-y_1 + y_2)$$
$$\dot{y}_2 = 28y_1 - y_2 - y_1y_2 + Cx_2^2$$
$$\dot{y}_3 = y_1y_2 - \frac{8}{3}y_3$$



Quiroga RQ, Arnhold J, Grassberger P. Learning driver-response relationships from synchronization patterns. Phys. Rev. E. 61, 5142, 2000

measuring interactions — state-space

identifying generalized synchronization



$$\psi = ?$$

measuring interactions state-space

identifying generalized synchronization from time series

when assuming existence of functional ψ

- project attractors onto some (joint) plane (look for GS)
- mutual false nearest neighbors
  (test for smoothness, continuity)
- epsilon-delta-statistics (*)
  (test for continuity, invertibility, differentiability, rang invariance)
- mutual nonlinear prediction (*)

when not assuming existence of functional ψ

- nonlinear interdependencies

(*) not discussed here; see LM Pecora et al., Phys Rev E 52, 3420, 1995 and SJ Schiff et al, Phys Rev E 54, 6708, 1996

measuring interactions **state-space**

identifying generalized synchronization from time series

look for GS

example: strongly, unidirectionally coupled Rössler systems (large c)

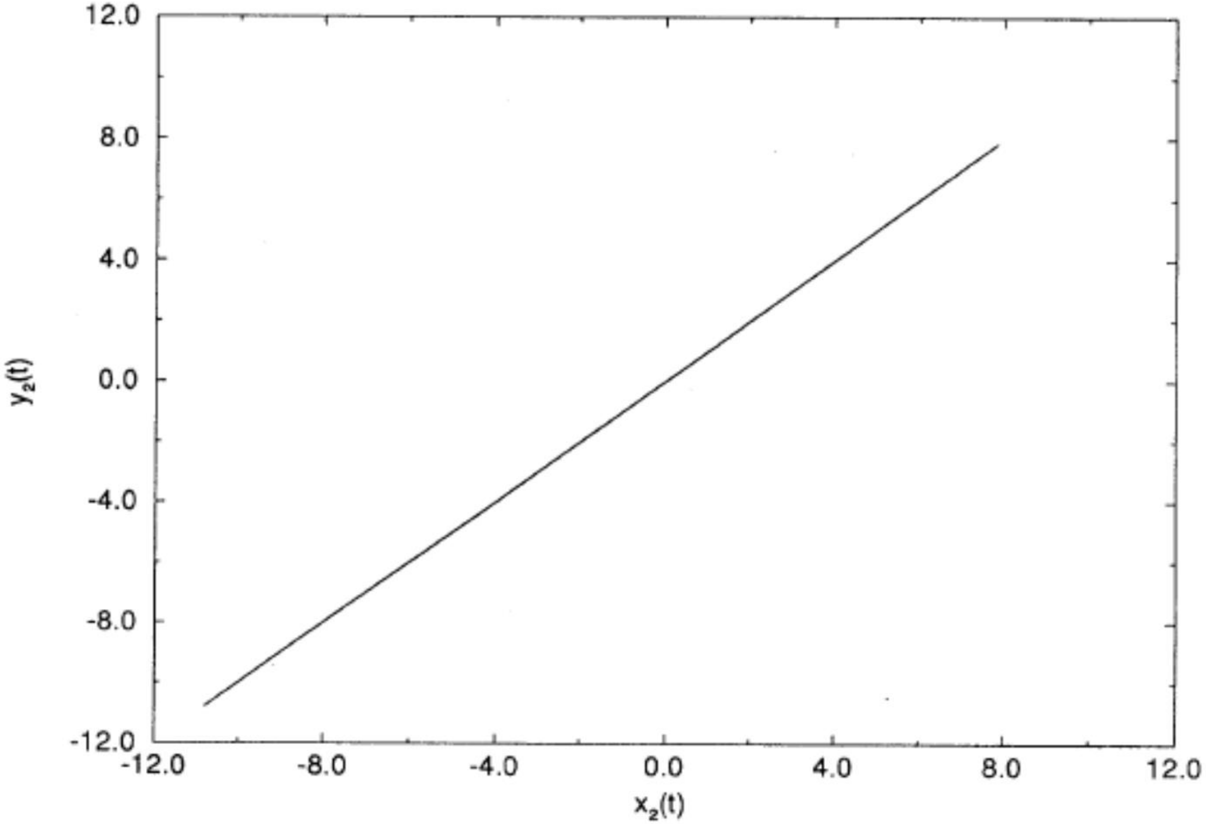
→ complete GS → $\psi = \mathrm{id}$

driver system:

$$\begin{aligned}\dot{x}_1 &= (x_2 + x_3)\\ \dot{x}_2 &= x_1 + 0.2x_2\\ \dot{x}_3 &= 0.2 + x_3(x_1 - 5.7)\end{aligned}$$

responder system:

$$\begin{aligned}\dot{y}_1 &= (y_2 + y_3) - c(y_1 - x_1)\\ \dot{y}_2 &= y_1 + 0.2y_2\\ \dot{y}_3 &= 0.2 + y_3(y_1 - 5.7)\end{aligned}$$



from: Rulkov NF, Sushchik MM, Tsimring LS, Abarbanel HD. Generalized synchronization of chaos in directionally coupled chaotic systems. Phys. Rev E 51, 980, 1995

measuring interactions **state-space**

identifying generalized synchronization from time series

look for GS

example: weakly, unidirectionally coupled Rössler systems (small c)
→ incomplete GS → $\psi \approx \mathrm{id}$ (?)

driver system:

$$\begin{aligned}\dot{x}_1 &= (x_2 + x_3)\\ \dot{x}_2 &= x_1 + 0.2x_2\\ \dot{x}_3 &= 0.2 + x_3(x_1 - 5.7)\end{aligned}$$

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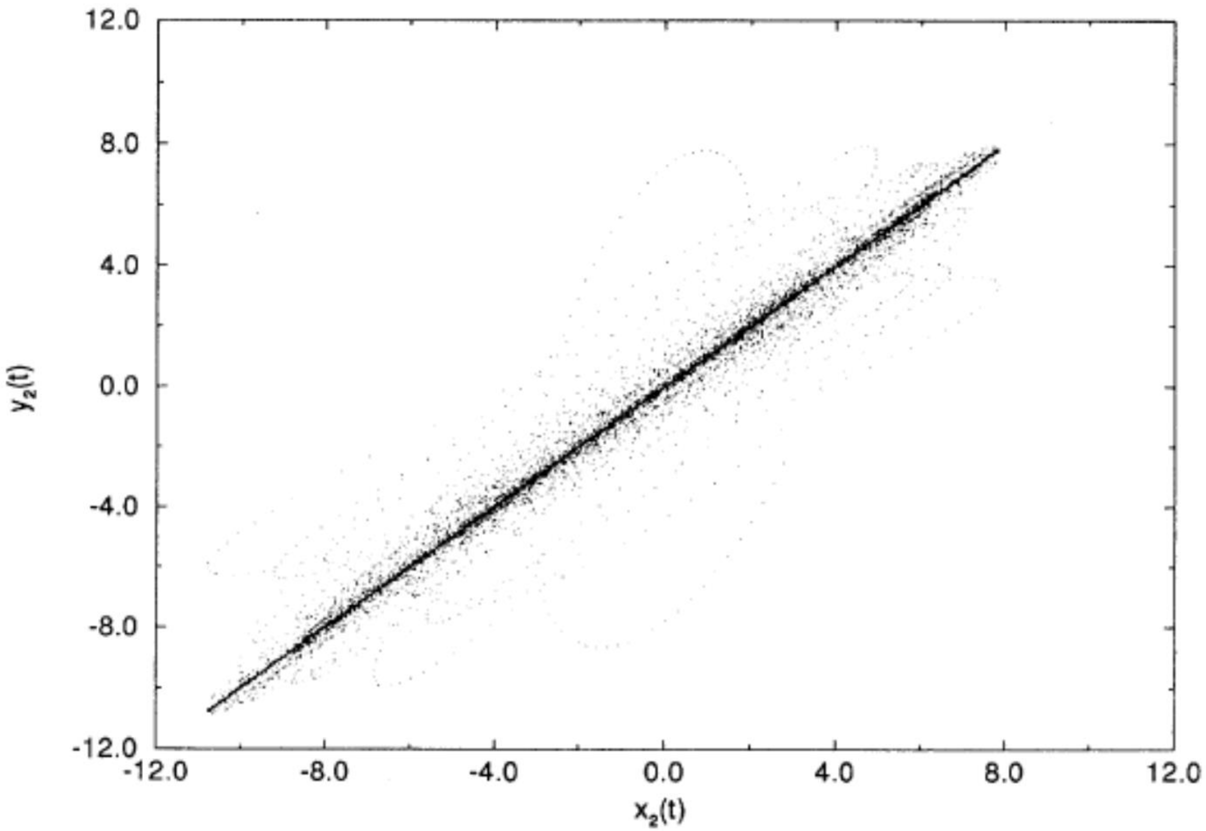

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measuring interactions **state-space**

identifying generalized synchronization from time series

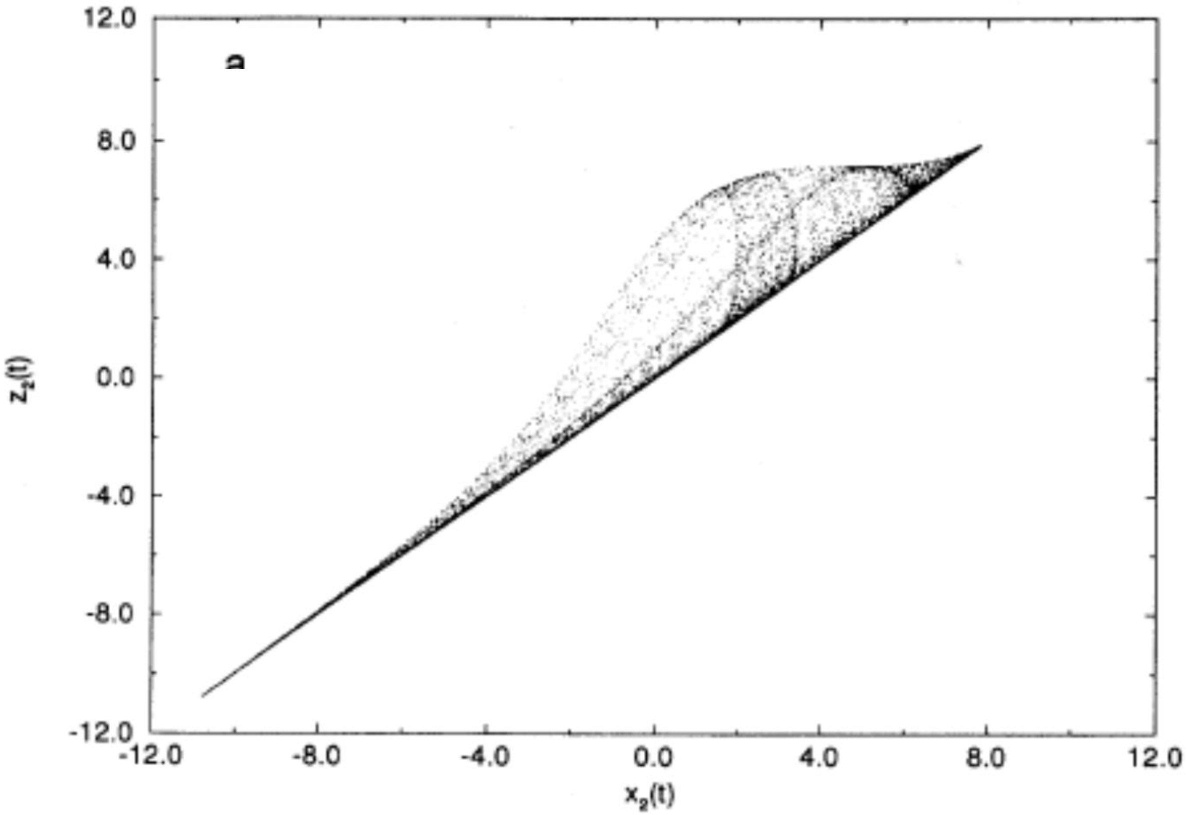
look for GS

example: strongly, unidirectionally coupled Rössler systems (large c) and some nonlinear transformation of responder system

→ complete GS → $\psi = ?$

new responder variables:

$$z_1 = y_1$$
$$z_2 = y_2 + 0.4 y_3 - 0.008 y_3^2$$
$$z_3 = y_3$$



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measuring interactions **state-space**

identifying generalized synchronization from time series

look for GS

example: weakly, unidirectionally coupled Rössler systems (small c) and some nonlinear transformation of responder system

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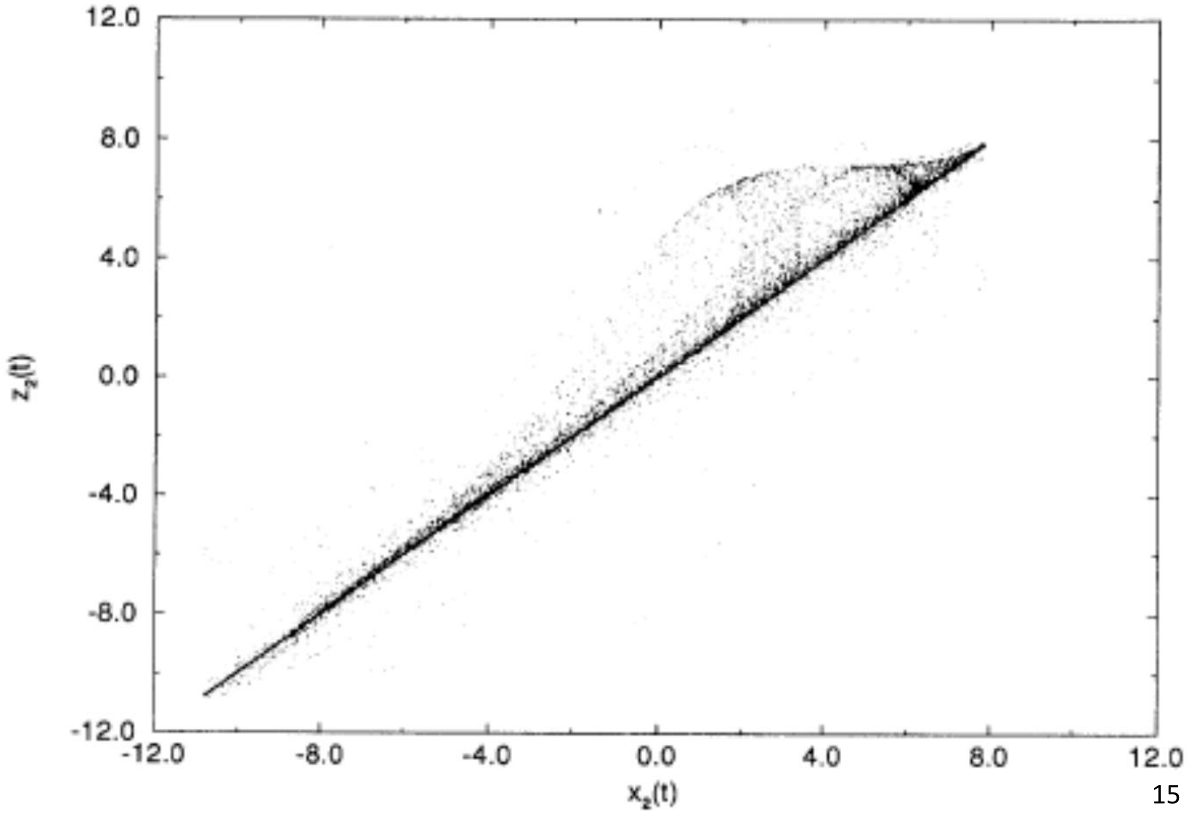

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measuring interactions **state-space**

identifying generalized synchronization from time series

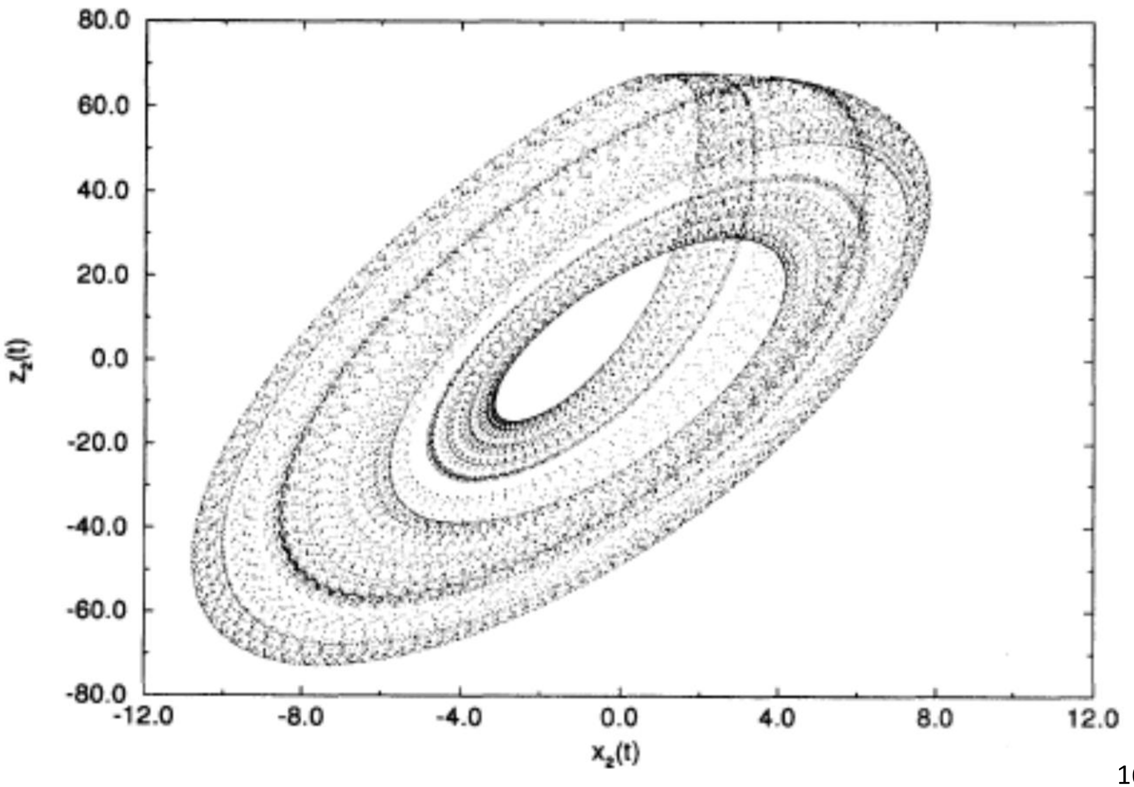
look for GS

example: strongly, unidirectionally coupled Rössler systems (large c) and some nonlinear transformation of delayed responder system

→ complete GS → $\psi = ?$

new responder variables:

$$
\begin{aligned}
z_1 &= y_1 \\
z_2 &= y_2 + 10 \int_{-\infty}^{t} \mathrm{e}^{-(t-\tau)} y_1(\tau) \mathrm{d}\tau \\
z_3 &= y_3
\end{aligned}
$$



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measuring interactions **state-space**

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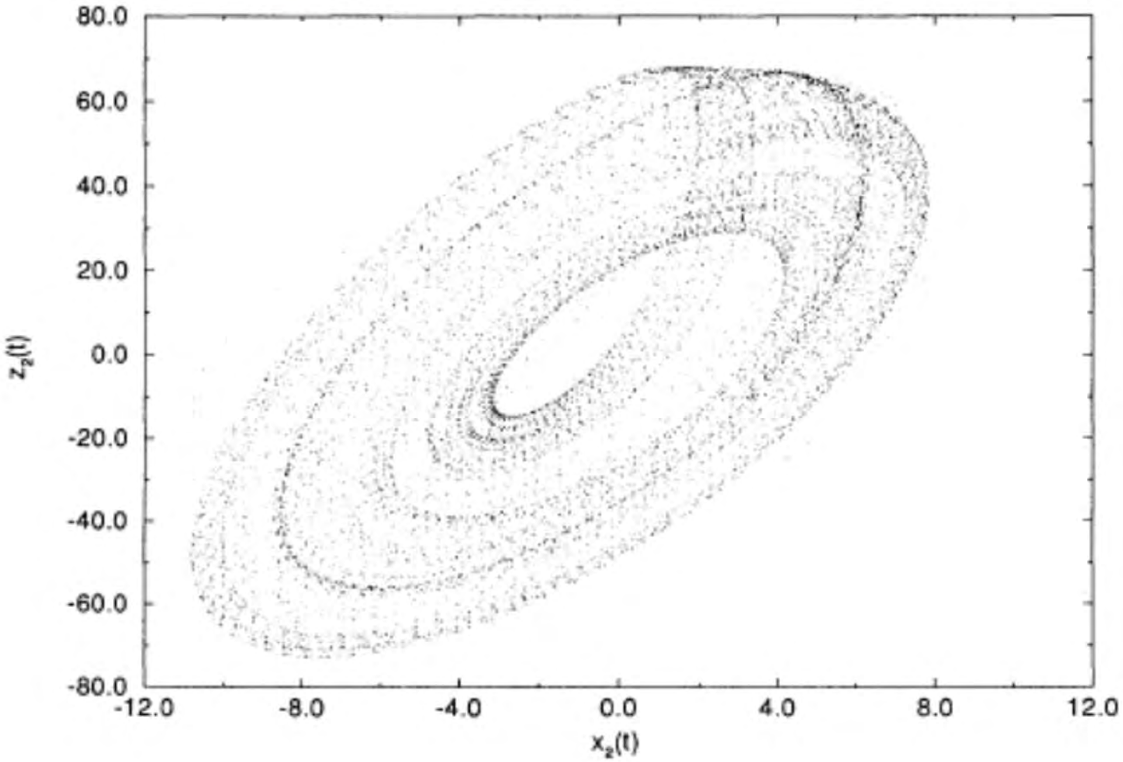

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measuring interactions **state-space**

identifying generalized synchronization from time series

look for GS

projecting attractors to some (joint) plane

- can not sufficiently identify properties of ψ
- suitable for strong-coupling-limit and for some conditions only
- misinterpretations due to nonlinear or delayed couplings
- other confounders?

- need more appropriate ansatz

measuring interactions **state-space**

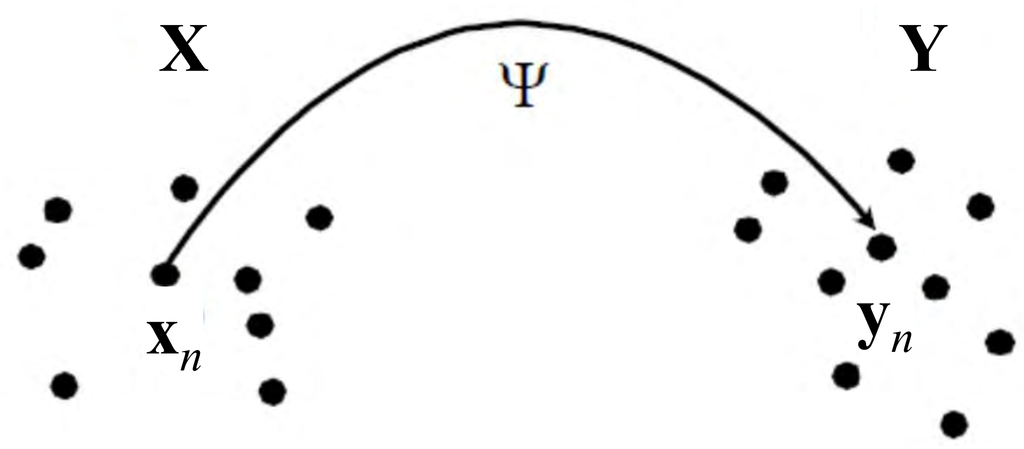
identifying generalized synchronization from time series

mutual false nearest neighbors

consider driver-responder system with observables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of driver X and responder Y

if $\mathbf{x}(t) = \Psi(\mathbf{y}(t))$ holds,
we find:

closeness in state-space of driver implies closeness in state-space of responder



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identifying generalized synchronization from time series

mutual false nearest neighbors

if $\mathbf{x}(t) = \Psi(\mathbf{y}(t))$ holds, we have ($\mathbf{D}$ is Jacobian matrix)

$$\mathbf{x}_n - \mathbf{x}_{n_{\mathrm{NND}}} = \Psi(\mathbf{y}_n) - \Psi(\mathbf{y}_{n_{\mathrm{NND}}}) \approx \mathbf{D}\Psi(\mathbf{y}_n)(\mathbf{y}_n - \mathbf{y}_{n_{\mathrm{NND}}})$$

and

$$\mathbf{x}_n - \mathbf{x}_{n_{\mathrm{NNR}}} = \Psi(\mathbf{y}_n) - \Psi(\mathbf{y}_{n_{\mathrm{NNR}}}) \approx \mathbf{D}\Psi(\mathbf{y}_n)(\mathbf{y}_n - \mathbf{y}_{n_{\mathrm{NNR}}})$$

NND (NNR) denote the number of nearest neighbors of state vector $\mathbf{x}_n = \mathbf{x}(t_n)$ ($\mathbf{y}$ analogous) of driver (responder) system

need (appropriately normalized) parameter that quantifies deviation from the above assumption

from: Rulkov NF, Sushchik MM, Tsimring LS, Abarbanel HD. Generalized synchronization of chaos in directionally coupled chaotic systems. Phys. Rev E 51, 980, 1995

measuring interactions **state-space**

identifying generalized synchronization from time series

mutual false nearest neighbors

$$P_{\mathrm{MFNN}}(n) := \frac{\mathbf{x}_n - \mathbf{x}_{n_{\mathrm{NND}}}}{\mathbf{y}_n - \mathbf{y}_{n_{\mathrm{NND}}}} \frac{\mathbf{y}_n - \mathbf{y}_{n_{\mathrm{NNR}}}}{\mathbf{x}_n - \mathbf{x}_{n_{\mathrm{NNR}}}} \longmapsto \begin{cases} \simeq 1: & \text{if complete GS} \\ \gg 1: & \text{else} \end{cases}$$

- appropriate delay-embedding
- nearest neighbors from closest distance to reference state
- average over (sufficiently) many reference states

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identifying generalized synchronization from time series

mutual false nearest neighbors

example: unidirectionally coupled Rössler systems and some nonlinear transformation of responder system

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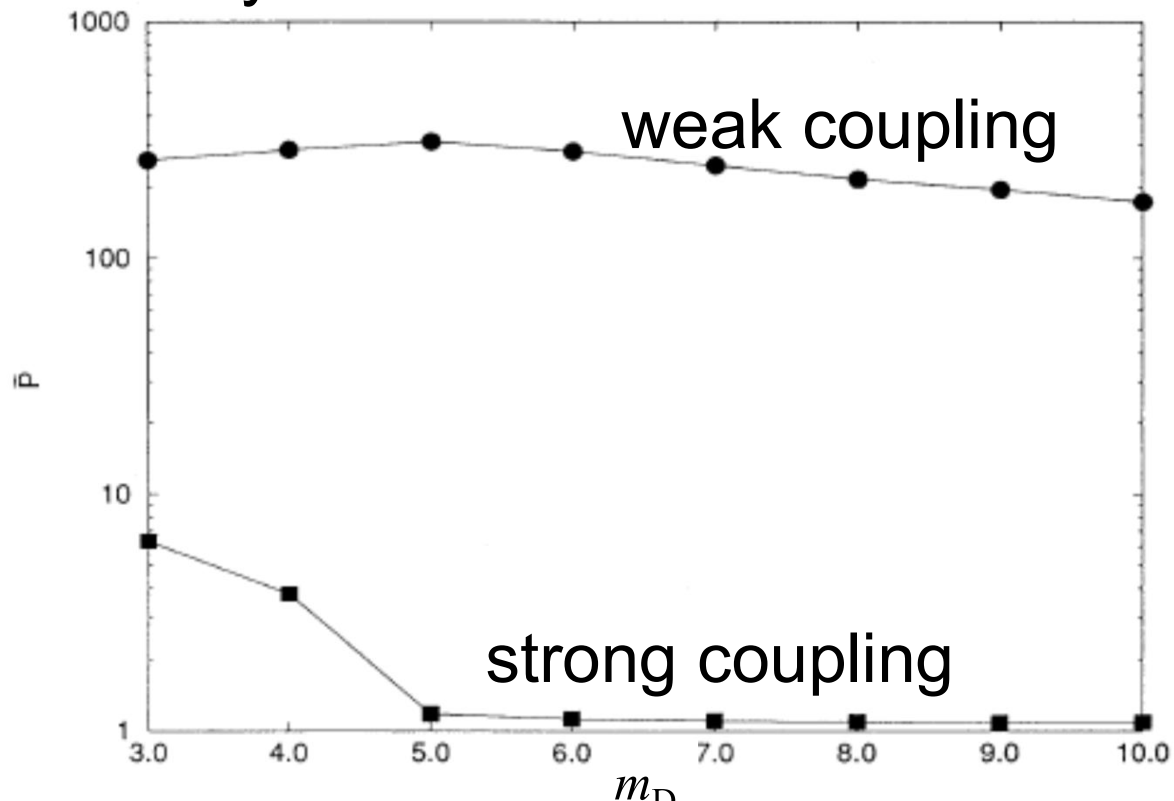

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measuring interactions **state-space**

identifying generalized synchronization from time series
mutual false nearest neighbors

example: unidirectionally coupled Rössler systems and some nonlinear transformation of delayed responder system

new responder variables:

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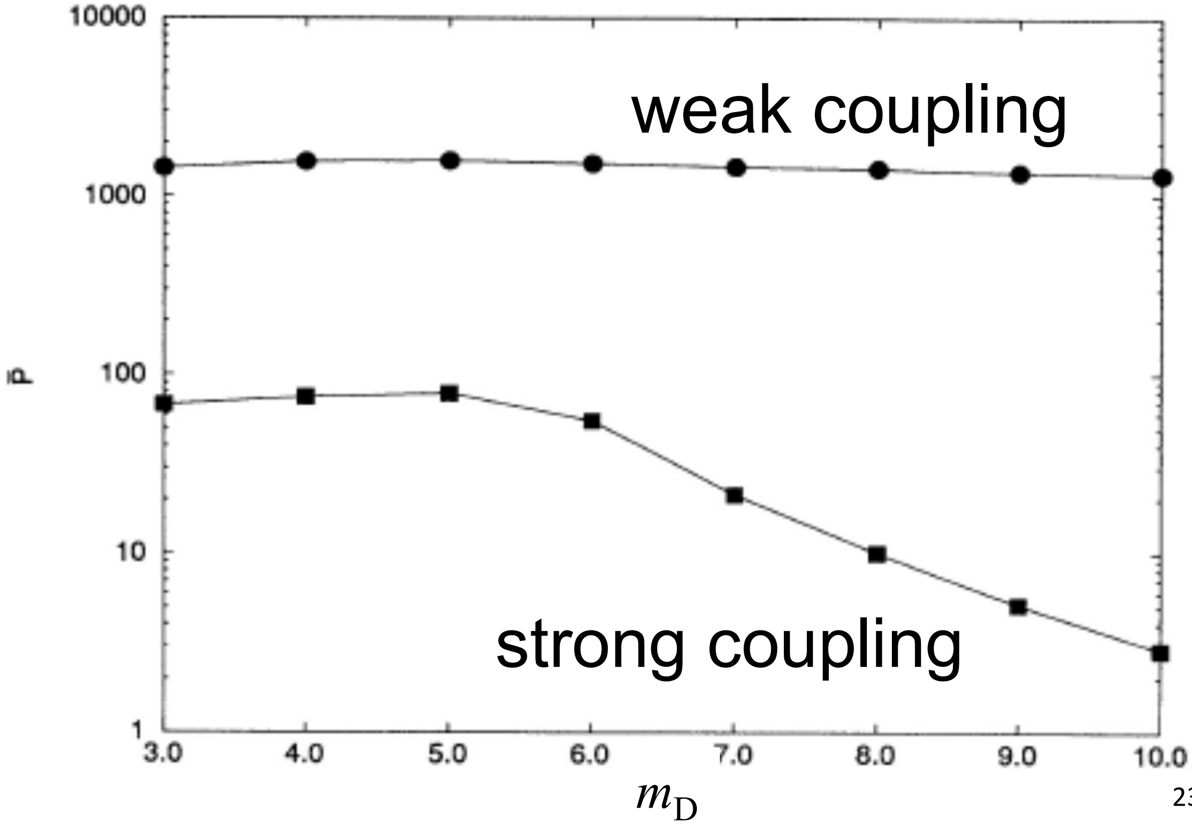

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measuring interactions **state-space**

identifying generalized synchronization from time series
mutual false nearest neighbors

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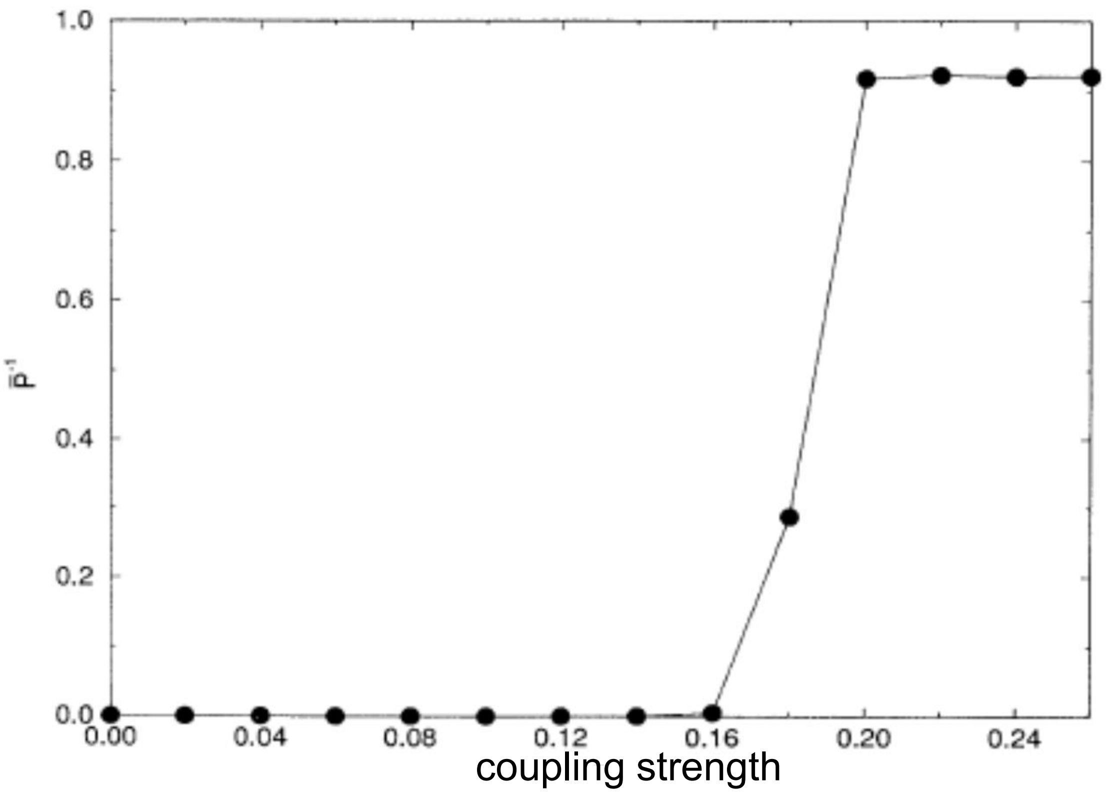

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measuring interactions **state-space**

identifying generalized synchronization from time series

mutual false nearest neighbors

P_{MFNN} increases with coupling strength

→ data-driven estimator for strength of interaction

symmetric ansatz

→ no indication for direction of interactions

ansatz assumes existence of ψ, well-defined properties of ψ (smooth, differentiable, invertible) for strong coupling only!

sensitive to noise, too many parameters

→ not well suited for time series analysis

measuring interactions **state-space**

identifying generalized synchronization from time series

nonlinear interdependence

Qs:

why assume existence of ψ? (necessary/relevant?)

weak coupling? (is more interesting case)

w.r.t. time series analysis: influence of noise?

existence of (unknown) third system, driving the others?

why assume determinism? (stochastic processes)

measuring interactions **state-space**

identifying generalized synchronization from time series
nonlinear interdependence

given observables $\mathbf{x}(t)$ and $\mathbf{y}(t)$ of systems X and Y

appropriate state-space reconstruction (time-delay embedding)

choose reference vectors $\mathbf{x}_n = \mathbf{x}(t_n)$ and $\mathbf{y}_n = \mathbf{y}(t_n)$

identify their k nearest neighbors (ε-environment, Euclidean distance) and denote their indices by $r_{n,j}$ and $s_{n,j}$, $j = 1, \ldots, k$

Arnhold J, Grassberger P, Lehnertz K, Elger CE. A robust method for detecting interdependences: application to intracranially recorded EEG. Physica D 134,:419, 1999

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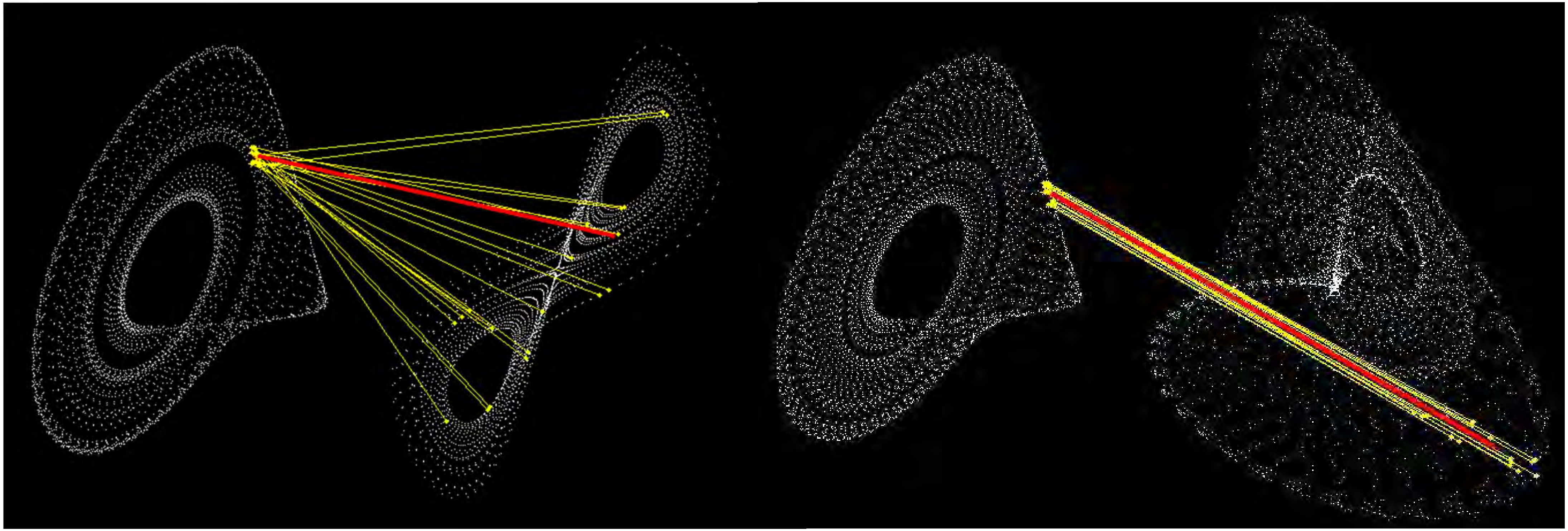
identifying generalized synchronization from time series

nonlinear interdependence

uncoupled strong coupling



reference vectors connected by red lines; neighbors connected by yellow lines

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identifying generalized synchronization from time series

nonlinear interdependence

define "true" and "false" distances for system X (analogously for Y)

$$R_n^{(k)}(\mathbf{X}) := \frac{1}{k}\sum_{j=1}^{k}\left(\mathbf{x}_n - \mathbf{x}_{r_{n,j}}\right)^2 \quad R_n^{(k)}(\mathbf{X}|\mathbf{Y}) := \frac{1}{k}\sum_{j=1}^{k}\left(\mathbf{x}_n - \mathbf{x}_{s_{n,j}}\right)^2$$

observations:

if X and Y strongly related: $R_n^{(k)}(\mathbf{X}) \simeq R_n^{(k)}(\mathbf{X}|\mathbf{Y})$

if X and Y independent: $R_n^{(k)}(\mathbf{X}) \ll R_n^{(k)}(\mathbf{X}|\mathbf{Y})$

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measuring interactions **state-space**

identifying generalized synchronization from time series
nonlinear interdependence

define “local” and “global” interdependence measure for system X (analogously for Y)

$$S_n^{(k)}(\mathbf{X}|\mathbf{Y}) := \frac{R_n^{(k)}(\mathbf{X})}{R_n^{(k)}(\mathbf{X}|\mathbf{Y})} \qquad S^{(k)}(\mathbf{X}|\mathbf{Y}) := \frac{1}{N}\sum_{n=1}^{N} S_n^{(k)}(\mathbf{X}|\mathbf{Y})$$

- measures confined to unit interval ($S=1$ strong interdependence)
- asymmetry $S(\mathbf{X}|\mathbf{Y}) \neq S(\mathbf{Y}|\mathbf{X})$ reflects different levels of complexity of systems

- no claims about causality → active-passive relationship

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measuring interactions **state-space**

identifying generalized synchronization from time series

nonlinear interdependence

alternatives and extensions

$$H(\mathbf{X}|\mathbf{Y}) := \frac{1}{N}\sum_{n=1}^{N} \log \frac{R_n^{(k)}(\mathbf{X})}{R_n^{(k)}(\mathbf{X}|\mathbf{Y})}$$

$$M(\mathbf{X}|\mathbf{Y}) := \frac{1}{N}\sum_{n=1}^{N} \frac{R_n(\mathbf{X}) - R_n^{(k)}(\mathbf{X}|\mathbf{Y})}{R_n(\mathbf{X}) - R_n^{(k)}(\mathbf{X})}$$

$$L(\mathbf{X}|\mathbf{Y}) := \frac{1}{N}\sum_{n=1}^{N} \frac{G_n(\mathbf{X}) - G_n^{(k)}(\mathbf{X}|\mathbf{Y})}{G_n(\mathbf{X}) - G_n^{(k)}(\mathbf{X})}$$

with the mean and minimal mean rank

$$G_n(\mathbf{X}) = \frac{N}{2} \text{ and } G_n^{(k)}(\mathbf{X}) = \frac{k+1}{2}$$

J. Arnhold, P. Grassberger, K. Lehnertz, C.E. Elger , Physica D 134,:419, 1999; R.G. Andrzejak, A. Kraskov, H. Stögbauer, F. Mormann, T. Kreuz, Phys. Rev. E 68, 066202, 2003; D. Chicharro, R. G. Andrzejak, Phys. Rev. E 80, 026217, 2009

measuring interactions **state-space**

identifying generalized synchronization from time series

nonlinear interdependence

symmetric and antisymmetric estimators for strength and direction of interaction:

$$A_+^{(k)}(\mathbf{X}|\mathbf{Y}) = A_+^{(k)}(\mathbf{Y}|\mathbf{X}) := \frac{1}{2}\left(A^{(k)}(\mathbf{X}|\mathbf{Y}) + A^{(k)}(\mathbf{Y}|\mathbf{X})\right)$$

$$A_-^{(k)}(\mathbf{X}|\mathbf{Y}) = -A_-^{(k)}(\mathbf{Y}|\mathbf{X}) := \frac{1}{2}\left(A^{(k)}(\mathbf{X}|\mathbf{Y}) - A^{(k)}(\mathbf{Y}|\mathbf{X})\right)$$

measuring interactions state-space

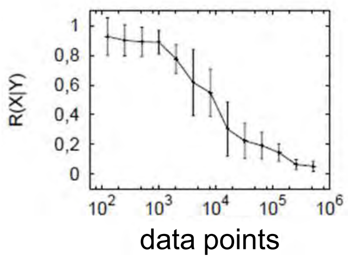
identifying generalized synchronization from time series

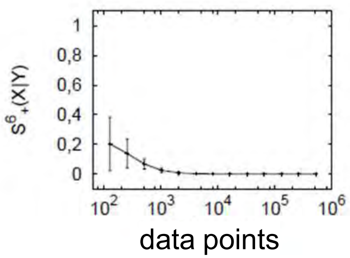
nonlinear interdependence

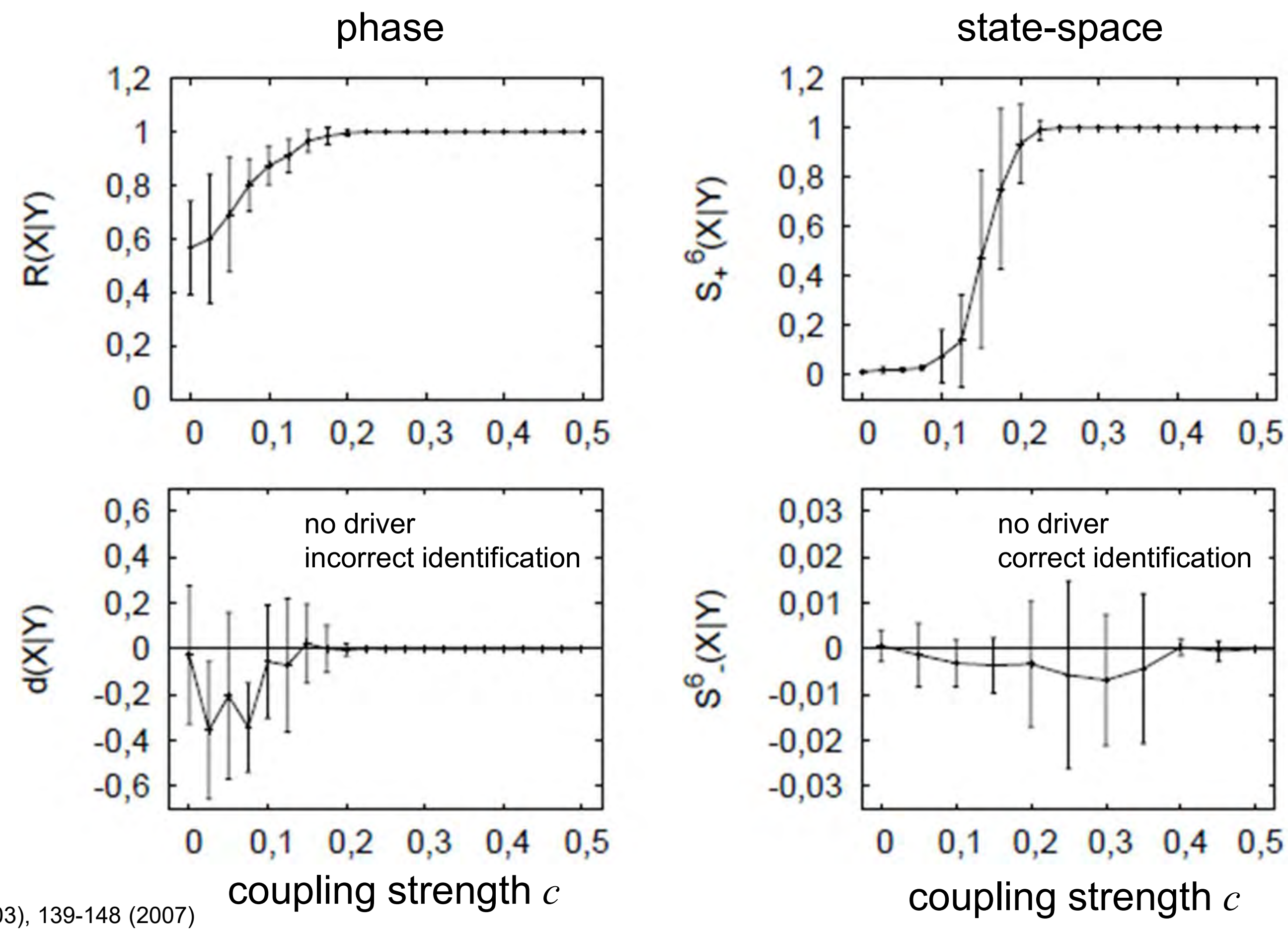
comparison with phase-based estimators

diffusively coupled Rössler oscillators
identical eigen-frequencies
50 realizations, 4096 data points
embedding dimension 5
embedding delay 1

$$\begin{aligned}
\dot{x}_{1,2} &= -\Omega_{1,2} y_{1,2} - z_{1,2} + c(x_{2,1} - x_{1,2}) \\
\dot{y}_{1,2} &= \Omega_{1,2} x_{1,2} + 0.165 y_{1,2} \\
\dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10) \\
\Omega_1 &= \Omega_2 = 0.89
\end{aligned}$$







Osterhage, H., Mormann, F., Wagner, T., & Lehnertz, K. Int. J. Neural Syst., 17(03), 139-148 (2007)

measuring interactions state-space

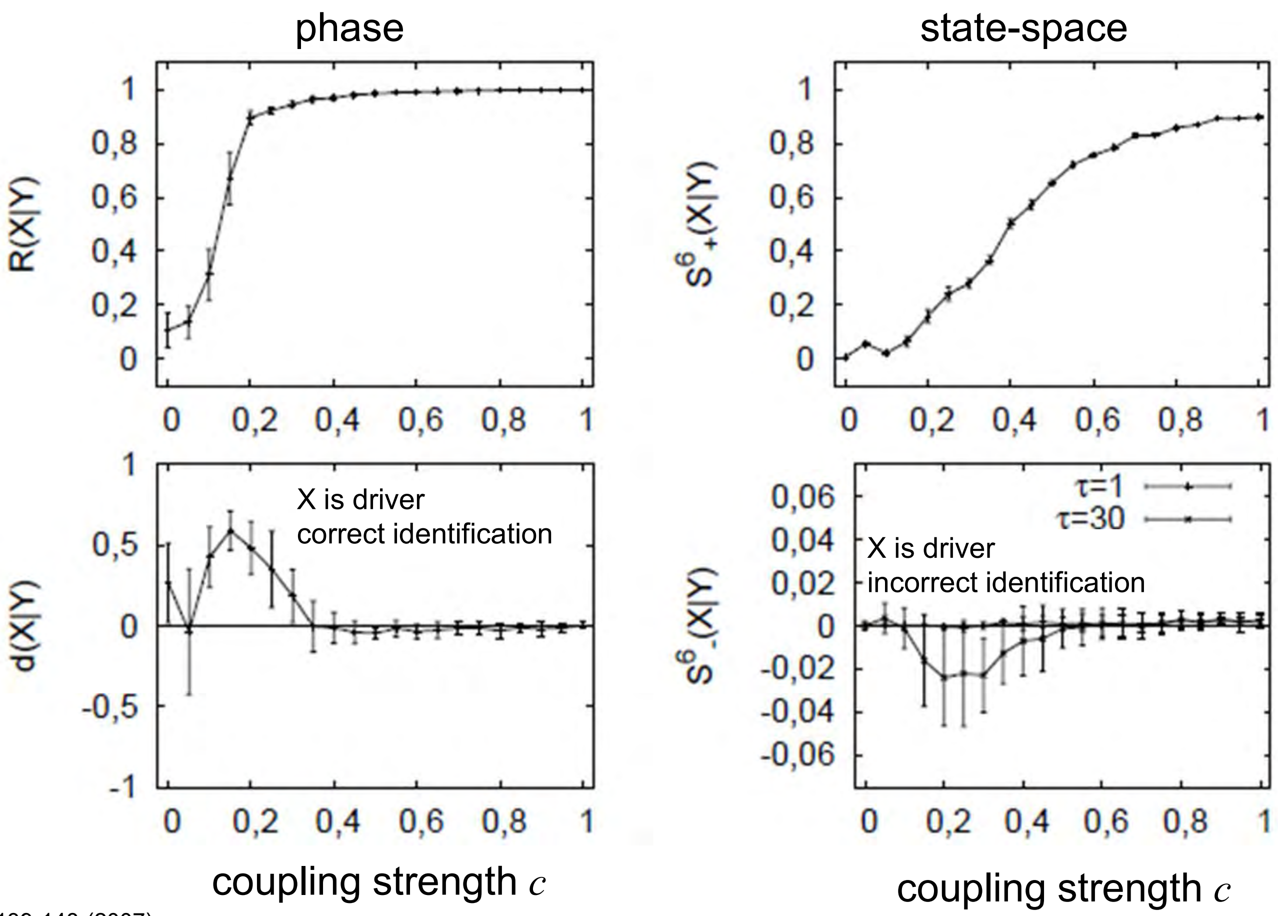
identifying generalized synchronization from time series

nonlinear interdependence

comparison with phase-based estimators

diffusively coupled Rössler oscillators
different eigen-frequencies
50 realizations, 4096 data points
embedding dimension 7
embedding delays (1, 30)

$$
\begin{aligned}
\dot{x}_{1,2} &= -\Omega_{1,2}y_{1,2} - z_{1,2} + c(x_{2,1} - x_{1,2}) \\
\dot{y}_{1,2} &= \Omega_{1,2}x_{1,2} + 0.165y_{1,2} \\
\dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10) \\
\Omega_1 &= 0.89 \;\; \Omega_2 = 0.95
\end{aligned}
$$



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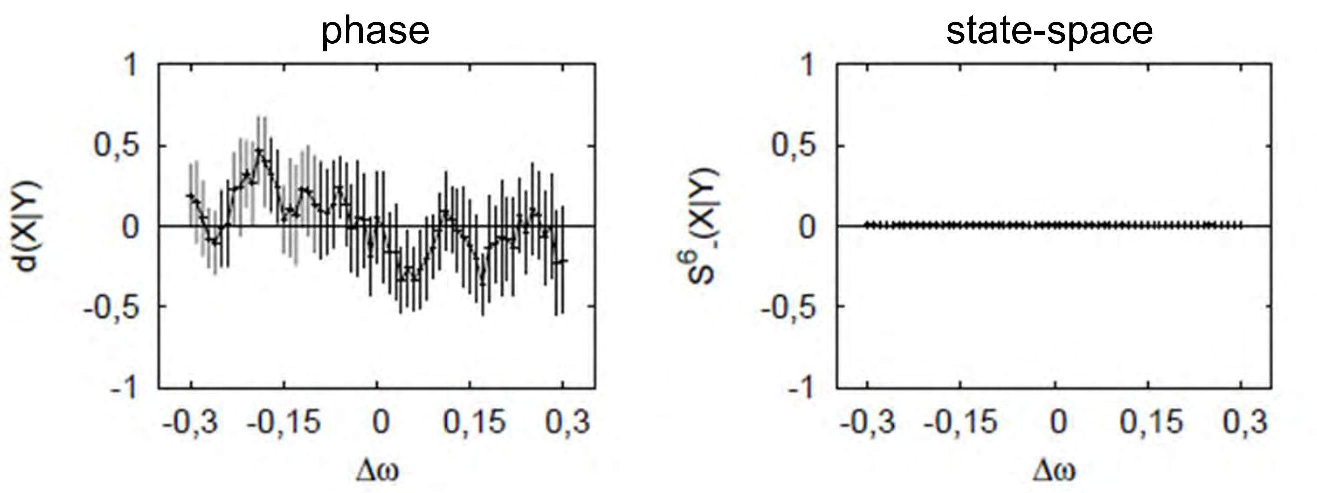
measuring interactions state-space

identifying generalized synchronization from time series
nonlinear interdependence
comparison with phase-based estimators

diffusively coupled Rössler oscillators
different eigen-frequencies
50 realizations, 4096 data points
embedding dimension 7
embedding delays (1, 30)

impact of frequency mismatch



phase-based:
the fast system appears to drive the other system

state-space-based:
appears unaffected

Osterhage, H., Mormann, F., Wagner, T., & Lehnertz, K. Int. J. Neural Syst., 17(03), 139-148 (2007)

measuring interactions **state-space**

identifying generalized synchronization from time series

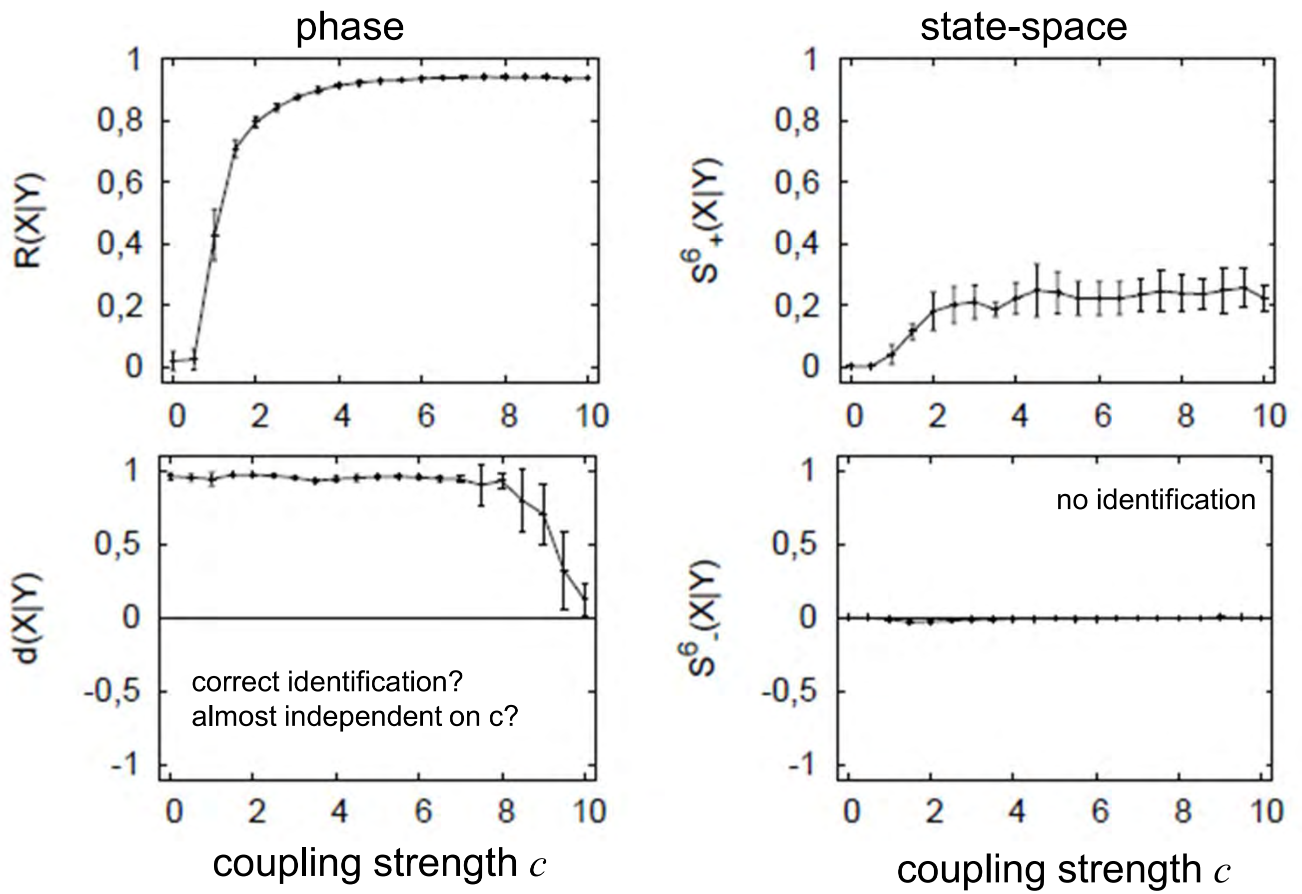
nonlinear interdependence

comparison with phase-based estimators

diffusively coupled Rössler (driver) – Lorenz (responder) oscillators
50 realizations, 16384 data points
embedding dimension 7
embedding delay 1

$$\begin{aligned}\dot{x}^{\mathrm{R}} &= -\Omega^{\mathrm{R}}(y^{\mathrm{R}} - z^{\mathrm{R}}) \\ \dot{y}^{\mathrm{R}} &= \Omega^{\mathrm{R}}(x^{\mathrm{R}} + 0.2y^{\mathrm{R}}) \\ \dot{z}^{\mathrm{R}} &= \Omega^{\mathrm{R}}\left(0.2 + z^{\mathrm{R}}(x^{\mathrm{R}} - 5.7)\right)\end{aligned}$$

$$\begin{aligned}\dot{x}^{\mathrm{L}} &= 10(y^{\mathrm{L}} - x^{\mathrm{L}}) \\ \dot{y}^{\mathrm{L}} &= \Omega^{\mathrm{L}}x^{\mathrm{L}} - y^{\mathrm{L}} - x^{\mathrm{L}}z^{\mathrm{L}} + c(y^{\mathrm{R}})^2 \\ \dot{z}^{\mathrm{L}} &= x^{\mathrm{L}}y^{\mathrm{L}} - \frac{8}{3}z^{\mathrm{L}}\end{aligned}$$



Osterhage, H., Mormann, F., Wagner, T., & Lehnertz, K. Int. J. Neural Syst., 17(03), 139-148 (2007)

measuring interactions state-space

identifying generalized synchronization from time series
nonlinear interdependence

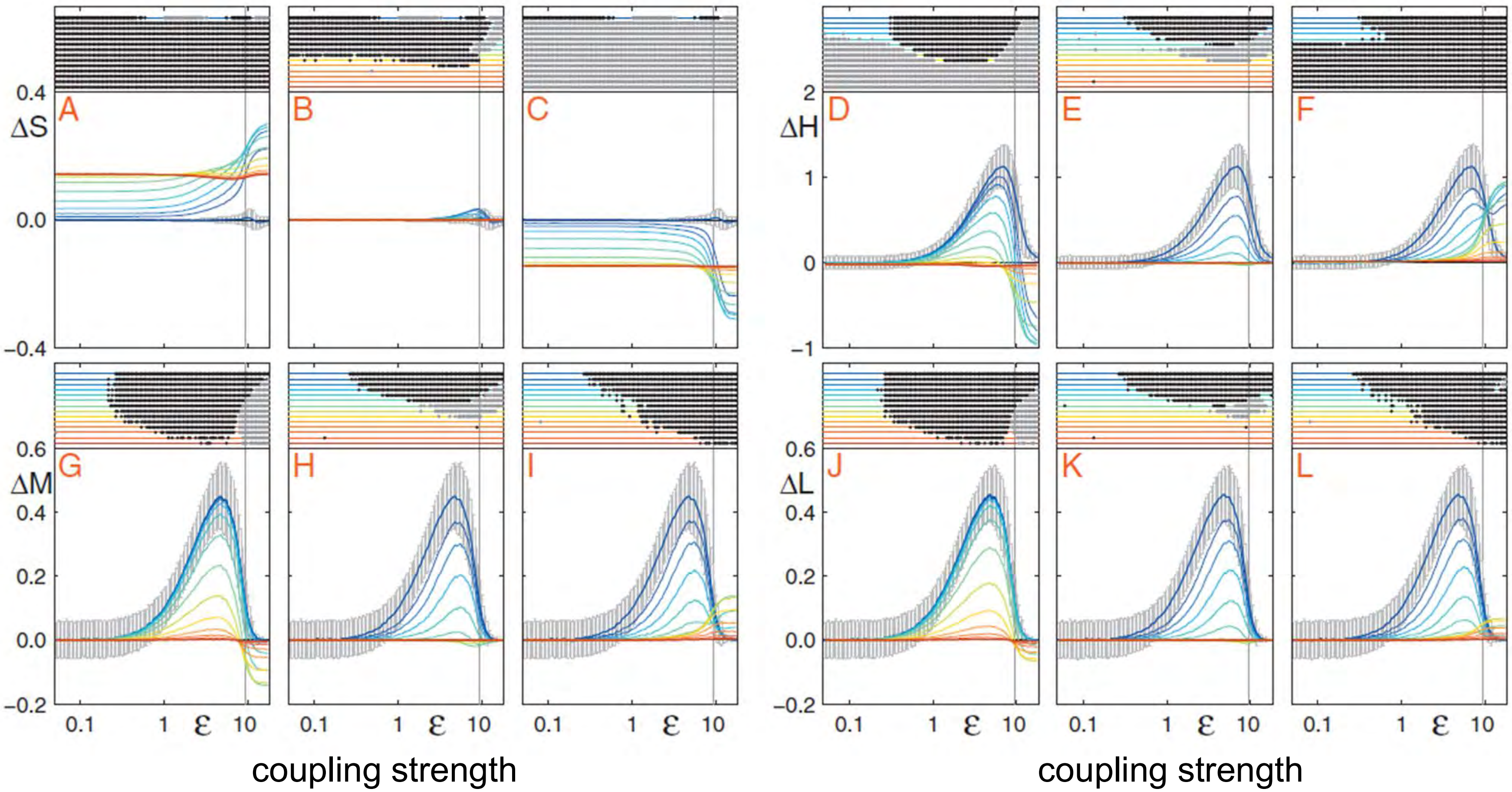
improved estimators (M and L)

diffusively coupled
Lorenz oscillators

1000 realizations
embedding dimension
+
embedding delay optimal
Theiler correction

different noise
contaminations
left: X only
middle: X and Y
right: Y only
(colors)

correct identification:
$\Delta A > 0$



D. Chicharro, R. G. Andrzejak, Phys. Rev. E 80, 026217, 2009

measuring interactions **state-space**

identifying generalized synchronization from time series

nonlinear interdependence

what can go wrong?

inappropriate normalization (translation, rotation, …)
all issues related to embedding
strongly coupled systems (direction of interaction)
not controlling for strong correlations in data
(see Theiler correction)

not accounting for different “complexities”
(eigen-frequencies, number of degrees of freedom, noise, …)

measuring interactions **state-space**

strength and direction of interaction

state-space-based estimators

advantages

- more general concept
- can capture all forms of synchronization
- high / moderate robustness against noise

disadvantages

- require appropriate choice of algorithmic parameter
- "faster" system → driver
  (need reliable surrogate test for directionality)
- may be fooled by (unobserved) third system

measuring interactions **state-space**

strength of interaction **comparison with other approaches**

test bed

coupled Hénon systems, Rössler oscillators, and Lorenz oscillators
optimally chosen algorithmic parameters
4096 data points
surrogate correction whenever necessary

measures:

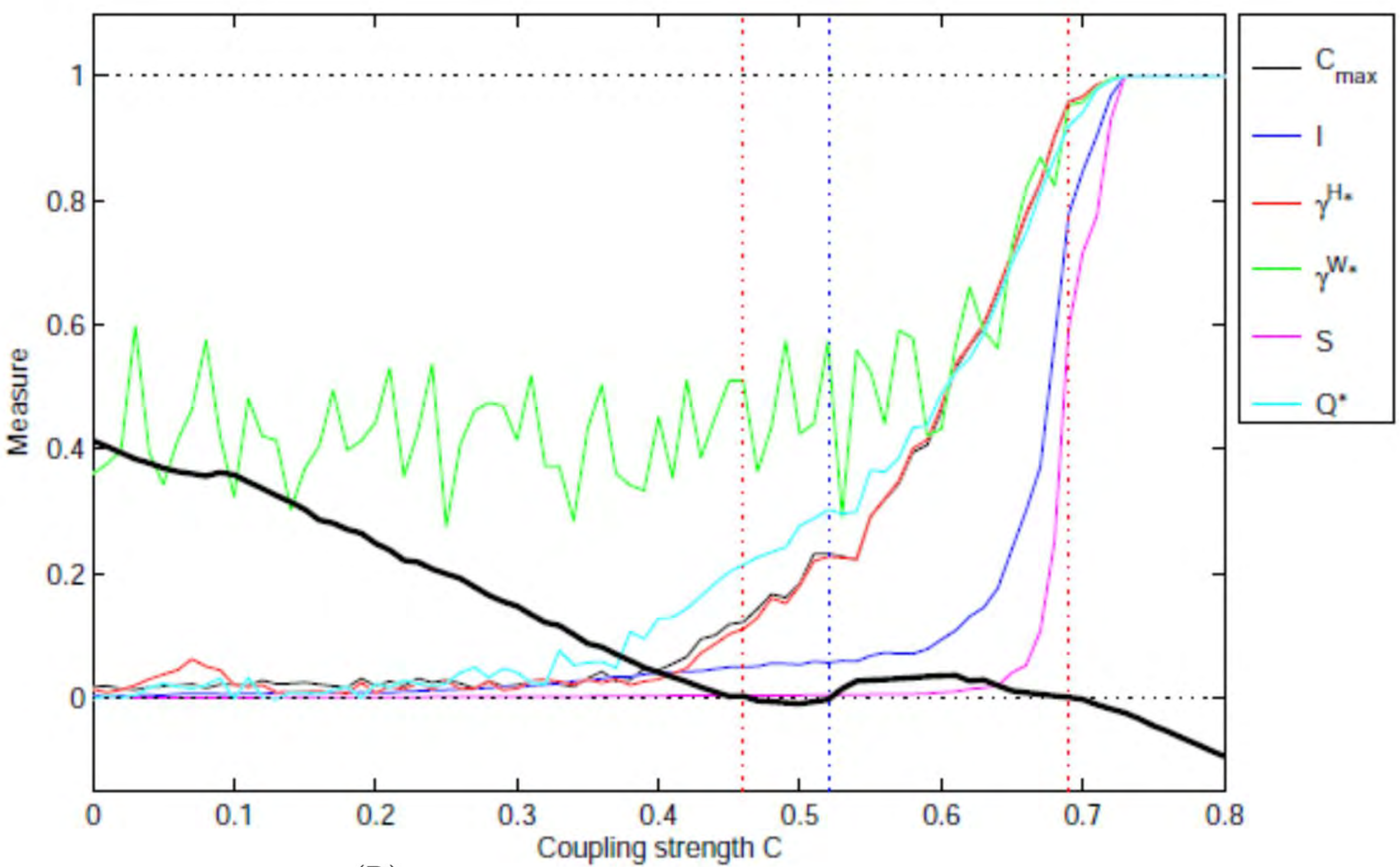
linear cross correlation
mutual information
phase-based approaches (Hilbert transform, wavelet transform)
nonlinear interdependency
event synchronization (see R.Q. Quiroga, T. Kreuz, P. Grassberger, Phys. Rev. E 66, 041904, 2002)

Kreuz T, Mormann F, Andrzejak RG, Kraskov A, Lehnertz K, Grassberger P. Physica D, 225, 29, 2007

measuring interactions state-space

strength of interaction comparison with other approaches

coupled Hénon systems



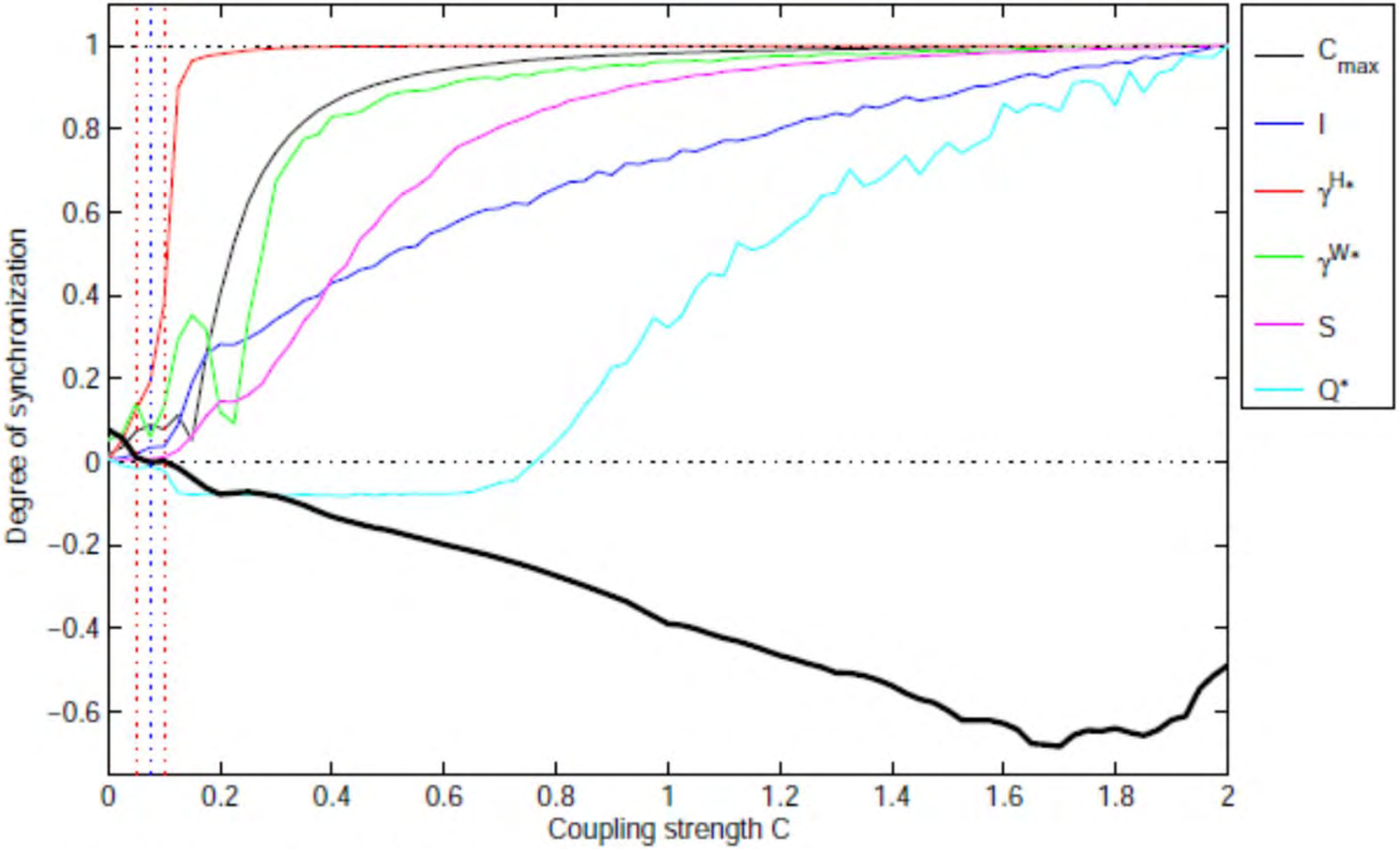
recall: generalized synchronization, iff $\lambda_i^{(\mathrm{R})} < 0 \ (\forall i)$

Kreuz T, Mormann F, Andrzejak RG, Kraskov A, Lehnertz K, Grassberger P. Physica D, 225, 29, 2007

measuring interactions **state-space**

strength of interaction **comparison with other approaches**

coupled Rössler oscillators



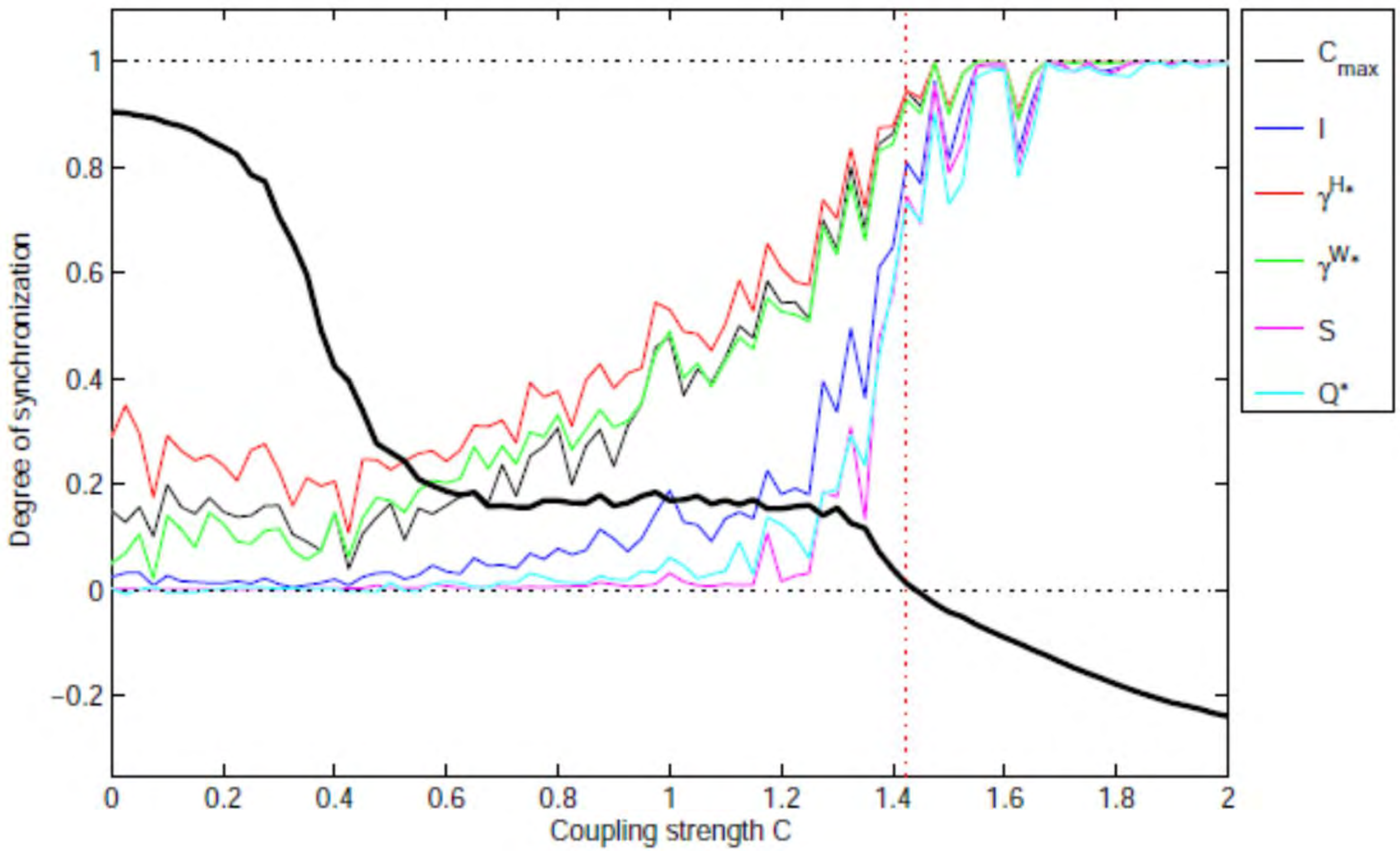
recall: generalized synchronization, iff $\lambda_i^{(\mathrm{R})} < 0 \ \ (\forall i)$

Kreuz T, Mormann F, Andrzejak RG, Kraskov A, Lehnertz K, Grassberger P. Physica D, 225, 29, 2007

measuring interactions — state-space

strength of interaction — comparison with other approaches

coupled Lorenz oscillators



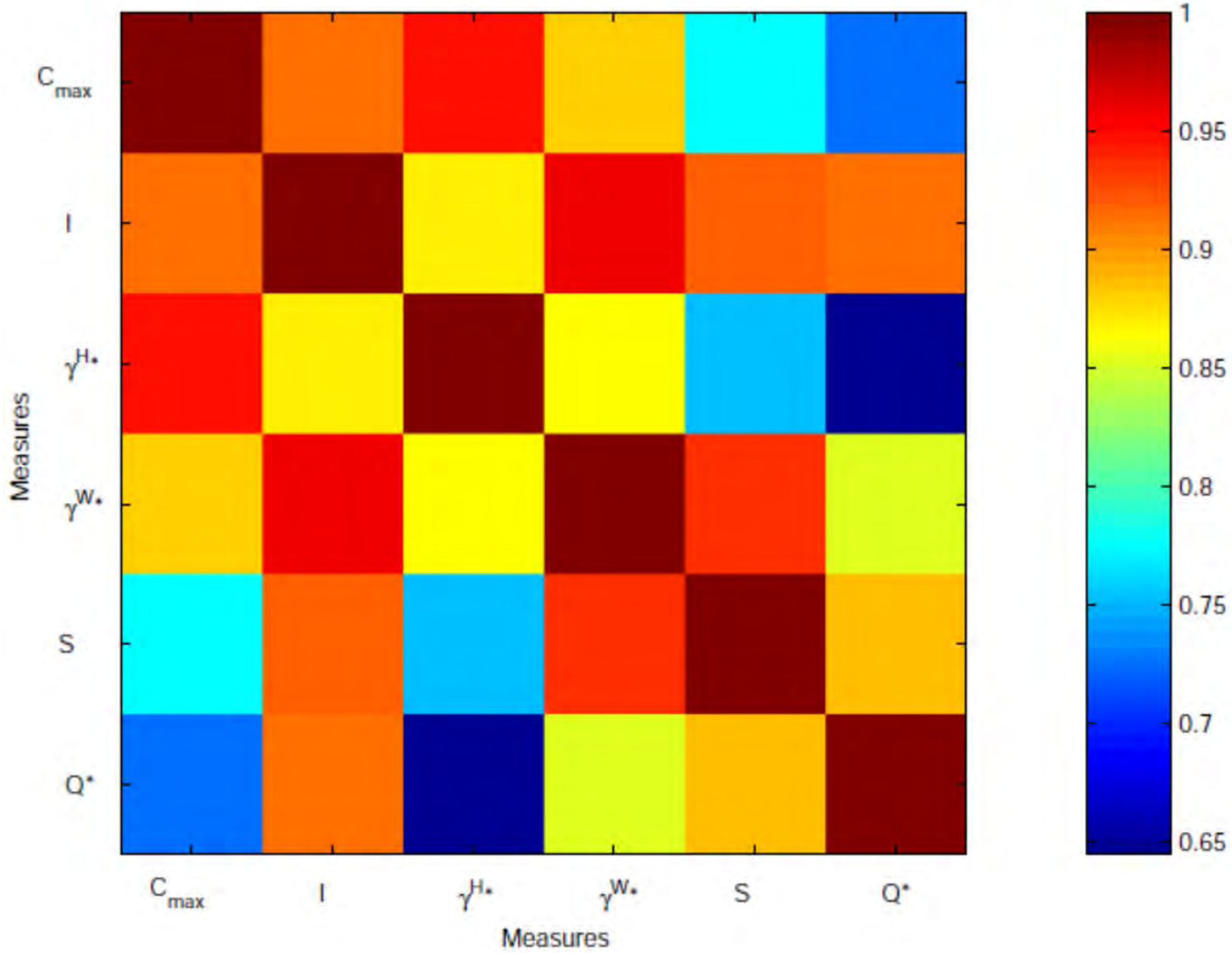
recall: generalized synchronization, iff $\lambda_i^{(R)} < 0 \ (\forall i)$

Kreuz T, Mormann F, Andrzejak RG, Kraskov A, Lehnertz K, Grassberger P. Physica D, 225, 29, 2007

measuring interactions **state-space**

strength of interaction **comparison with other approaches**

correlation between approaches



Kreuz T, Mormann F, Andrzejak RG, Kraskov A, Lehnertz K, Grassberger P. Physica D, 225, 29, 2007

measuring interactions **state-space**

strength of interaction **comparison with other approaches**

there is no “best approach”

dependent on specific application

choose approach according to quality and type of data

even combinations of approaches might be useful

Kreuz T, Mormann F, Andrzejak RG, Kraskov A, Lehnertz K, Grassberger P. Physica D, 225, 29, 2007

measuring interactions **state-space**

direction of interaction **comparison with other approaches**

test bed

coupled AR models, Hénon systems, Rössler-Lorenz oscillators, Rössler oscillators (large eigen-frequency mismatch), fishery model, two uncoupled systems driven by a third one

optimally chosen algorithmic parameters; 20.000 data points

surrogate correction whenever necessary

measures:

Granger causality (various approaches)

transfer entropy

nonlinear interdependency

predictability improvement (A. Krakovská and F. Hanzely, Phys. Rev. E 94, 052203, 2016)

Krakovská, A., Jakubík, J., Chvosteková, M., Coufal, D., Jajcay, N. Paluš, M., Phys. Rev. E, 97, 042207, 2018

measuring interactions **state-space**

direction of interaction **comparison with other approaches**

- results of different methods often contradicted each other
- methods differed considerably in their capability to reveal presence and direction of coupling and to distinguish causality from correlation
- outputs of methods difficult to compare
- low specificity was the problem of most methods
- choose the right method for a particular type of data
  "simple" cases → linear methods
  "complex" cases → information-theoretic and/or nonlinear methods

- blind application of any causality test easily leads to incorrect conclusions

Krakovská, A., Jakubík, J., Chvosteková, M., Coufal, D., Jajcay, N. Paluš, M., Phys. Rev. E, 97, 042207, 2018

measuring interactions **state-space**

other ideas (for the sake of completeness)

from (auto) correlation sum

$$C^{\mathrm{XX}}(\epsilon) := \frac{1}{N}\sum_i \left(\frac{1}{N}\sum_j \Theta\left(\epsilon - |\mathbf{x}_i - \mathbf{x}_j|\right)\right) \quad \text{with } C^{\mathrm{XX}}(\epsilon) \propto \epsilon^{D_2}$$

to cross-correlation sum

$$C^{\mathrm{XY}}(\epsilon) := \frac{1}{N}\sum_i \left(\frac{1}{N}\sum_j \Theta\left(\epsilon - |\mathbf{x}_i - \mathbf{y}_j|\right)\right) \quad \text{with } C^{\mathrm{XY}}(\epsilon) \propto \epsilon^{\min(\dim(\mathrm{X}\oplus\mathrm{Y}))}$$

H. Kantz. Quantifying the closeness of fractal measures. Phys. Rev. E, 49, 5091, 1994

measuring interactions

state-space

other ideas

cross-correlation sum

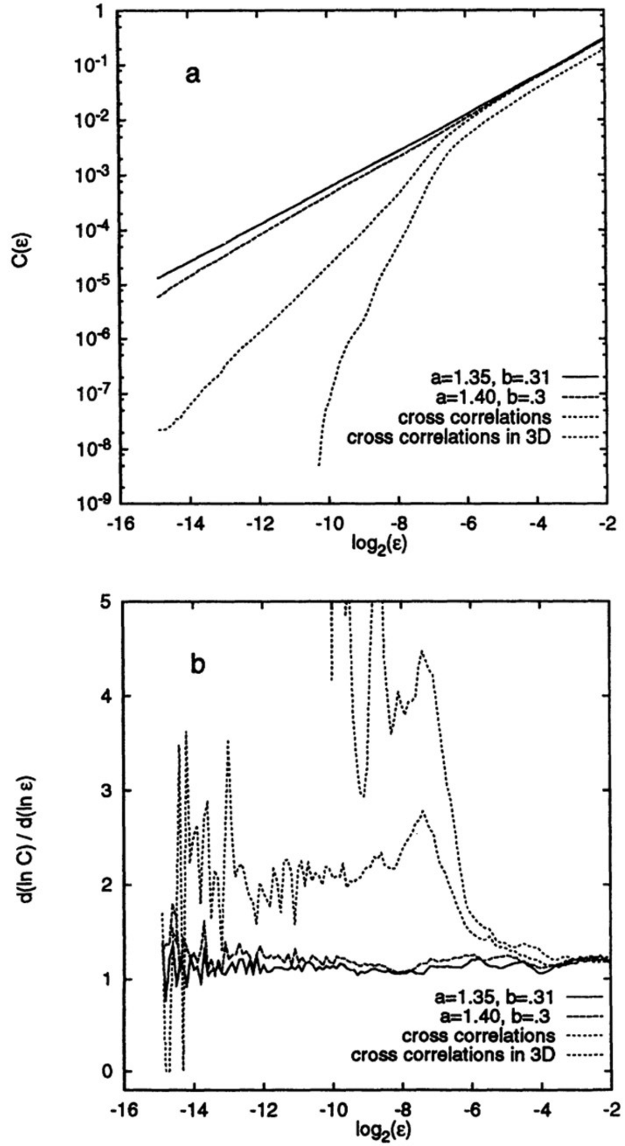
identical systems if:

$$C^{\mathrm{XX}}(\epsilon) = C^{\mathrm{YY}}(\epsilon) = C^{\mathrm{XY}}(\epsilon) \quad \forall \epsilon$$

need appropriate definition for "similarity"

no information about dynamics

not well suited for time series analysis



two Hénon systems with slightly different control parameters

H. Kantz. Quantifying the closeness of fractal measures. Phys. Rev. E, 49, 5091, 1994

measuring interactions **state-space**

other ideas

- dimension of interaction dynamics
- Hausdorff distance between attractors
- similarity of state-space densities (χ^2 test)
- similarity based on cross-predictability

→ no or only restricted information about dynamics
→ robustness, computational issues
→ do not provide information about strength and direction of interaction