Measuring Interactions
from Time Series

state-space*-based techniques

*same as phase-space but used here to avoid confusion with “phase”
Fundamentals of Analyzing Biomedical Signals

**Interactions**

measuring interactions

**state-space**

**basic idea:** interaction $\Leftrightarrow$ dynamics on attractor $\Leftrightarrow$ synchronization

- the “stronger” the interaction, the more similar are the attractors
- testing for conditional changes of attractor properties indexes directionality

more comprehensive characterization of interacting nonlinear systems

need to extend the classical concept of synchronization
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synchronization

- requires some (self-sustained) oscillatory behavior of autonomous systems

- does not necessarily indicate synchronous motion

- requires weak coupling
  strong coupling $\rightarrow$ identical motion $\rightarrow$ not of interest
  (natural phenomena $\Leftrightarrow$ weak coupling!)

- is a dynamical phenomenon, not a state!
measuring interactions

given observables $x(t)$ and $y(t)$ of systems $X$ and $Y$

**complete (or identical) synchronization**

$$\lim_{t \to \infty} |x(t) - y(t)| \equiv 0$$

**lag synchronization**

$$x(t + \tau) = y(t) \quad \forall t$$

**phase synchronization**

$$\Delta \phi_{XY}(t) := n\phi_X(t) - m\phi_Y(t) \leq \text{const}; \quad \forall t \text{ and } (n, m) \in \mathbb{N}$$
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complete synchronization:
    identical systems or infinitely strong coupling

lag synchronization:
    for $\tau \to 0$: cross-over to complete synchronization

phase synchronization:
    intuitively: phase sync. $\to$ lag sync. $\to$ complete sync.
    but: too many counterexamples

these concepts do not capture interactions between attractors
    $\to$ need another concept
measuring interactions

given observables \( x(t) \) and \( y(t) \) of systems \( X \) and \( Y \)

**generalized synchronization (GS)**

there is a functional \( \psi \) such that (after a transitory evolution from appropriately chose initial condition), we have:

\[
x(t) = \Psi(y(t))
\]

the dynamical state of one of the systems is completely determined by the state of the other

trajectory of joint system confined to sub-manifold of joint state-space

\[
\dim(X \oplus Y) \leq \dim(X) + \dim(Y)
\]

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generalized synchronization

- if systems are mutually coupled \( \psi \) has to be invertible
- for a driver-responder configuration \( \psi \) does not need to be invertible
- if \( \psi \) is the identity, we have identical synchronization

- one can find phase synchronization in case of generalized synchronization

but:
- generalized synchronization is not a necessary condition for phase synchronization

since 2000: attempts to find a unifying definition for synchronization

e.g. Brown R, Kocarev L. A unifying definition of synchronization for dynamical systems. Chaos 10, 344, 2000
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identifying generalized synchronization

- equations of motion are known a priori

- let $d$ and $r$ denote the dimensions of a driver (D) and a responder system (R), resp.

- let $\lambda_i^{(D)}, i = 1, \ldots, d$ denote the Lyapunov spectrum of the driver system, and $\lambda_i^{(R)}, i = 1, \ldots, r$ the spectrum of the responder system

- generalized synchronization, iff $\lambda_i^{(R)} < 0 \ (\forall i)$

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identifying generalized synchronization

driver: (modified) Rössler system

\[
\begin{align*}
\dot{x}_1 &= -\alpha(x_2 + x_3) \\
\dot{x}_2 &= \alpha(x_1 + 0.2x_2) \\
\dot{x}_3 &= \alpha(0.2 + x_3(x_1 - 5.7))
\end{align*}
\]

responder: Lorenz system

\[
\begin{align*}
\dot{y}_1 &= 10(-y_1 + y_2) \\
\dot{y}_2 &= 28y_1 - y_2 - y_1y_2 + Cx_2^2 \\
\dot{y}_3 &= y_1y_2 - \frac{8}{3}y_3
\end{align*}
\]

measuring interactions
identifying generalized synchronization

ψ = ?
measuring interactions

identifying generalized synchronization from time series

when assuming existence of functional $\psi$

- project attractors onto some (joint) plane (look for GS)
- mutual false nearest neighbors
  (test for smoothness, continuity)
- epsilon-delta-statistics (*)
  (test for continuity, invertibility, differentiability, rang invariance)
- mutual nonlinear prediction (*)

when not assuming existence of functional $\psi$

- nonlinear interdependencies

measuring interactions

identifying generalized synchronization from time series

look for GS

equation: strongly, unidirectionally coupled Rössler systems (large $c$) → complete GS → $\psi = \text{id}$

driver system:

\[
\begin{align*}
\dot{x}_1 &= (x_2 + x_3) \\
\dot{x}_2 &= x_1 + 0.2x_2 \\
\dot{x}_3 &= 0.2 + x_3(x_1 - 5.7)
\end{align*}
\]

responder system:

\[
\begin{align*}
\dot{y}_1 &= (y_2 + y_3) - c(y_1 - x_1) \\
\dot{y}_2 &= y_1 + 0.2y_2 \\
\dot{y}_3 &= 0.2 + y_3(y_1 - 5.7)
\end{align*}
\]

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identifying generalized synchronization from time series

look for GS

driver system:
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\end{align*}
\]

example: weakly, unidirectionally coupled Rössler systems (small \(c\)) → incomplete GS → \(\psi \approx \text{id}(?)\)

measuring interactions

identifying generalized synchronization from time series

look for GS

direct example: strongly, unidirectionally coupled Rössler systems (large $c$) and some nonlinear transformation of responder system

→ complete GS \( \psi = ? \)

new responder variables:

\[
\begin{align*}
z_1 &= y_1 \\
z_2 &= y_2 + 0.4y_3 - 0.008y_3^2 \\
z_3 &= y_3
\end{align*}
\]

measuring interactions

identifying generalized synchronization from time series

look for GS

example: weakly, unidirectionally coupled Rössler systems (small $c$) and some nonlinear transformation of responder system → incomplete GS → $\psi = ?$

new responder variables:

$z_1 = y_1$

$z_2 = y_2 + 0.4y_3 - 0.008y_3^2$

$z_3 = y_3$

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look for GS

e.g.: strongly, unidirectionally coupled Rössler systems (large $c$) and some nonlinear transformation of delayed responder system

$\rightarrow$ complete GS $\rightarrow \psi = ?$

new responder variables:

$$z_1 = y_1$$
$$z_2 = y_2 + 10 \int_{-\infty}^{t} e^{-(t-\tau)} y_1(\tau) d\tau$$
$$z_3 = y_3$$

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look for GS

element: weakly, unidirectionally coupled Rössler systems (small $c$) and some nonlinear transformation of delayed responder system $\rightarrow$ incomplete GS $\rightarrow$ $\psi = ?$

ew responder variables:

$$
\begin{align*}
    z_1 &= y_1 \\
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\end{align*}
$$

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identifying generalized synchronization from time series

projecting attractors to some (joint) plane

- can not sufficiently identify properties of $\psi$
- suitable for strong-coupling-limit and for some conditions only
- misinterpretations due to nonlinear or delayed couplings
- other confounders?

- need more appropriate ansatz
measuring interactions state-space
identifying generalized synchronization from time series mutual false nearest neighbors
consider driver-responder system with observables $x(t)$ and $y(t)$ of driver $X$ and responder $Y$

if $x(t) = \Psi(y(t))$ holds, we find:

**closeness in state-space of driver implies closeness in state-space of responder**
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mutual false nearest neighbors

if \( x(t) = \Psi(y(t)) \) holds, we have (\( D \) is Jacobian matrix)

\[
x_n - x_{n_{\text{NND}}} = \Psi(y_n) - \Psi(y_{n_{\text{NND}}}) \approx D\Psi(y_n)(y_n - y_{n_{\text{NND}}})
\]

and

\[
x_n - x_{n_{\text{NNR}}} = \Psi(y_n) - \Psi(y_{n_{\text{NNR}}}) \approx D\Psi(y_n)(y_n - y_{n_{\text{NNR}}})
\]

NND (NNR) denote the number of nearest neighbors of state vector \( x_n = x(t_n) \) (\( y \) analogous) of driver (responder) system

need (appropriately normalized) parameter that quantifies deviation from the above assumption

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mutual false nearest neighbors

\[ P_{MFNN}(n) := \frac{x_n - x_{n_{NND}}}{y_n - y_{n_{NND}}} \frac{y_n - y_{n_{NNR}}}{x_n - x_{n_{NNR}}} \]

\[ \begin{cases} \simeq 1 : \text{ if complete GS} \\ \gg 1 : \text{ else} \end{cases} \]

- appropriate delay-embedding
- nearest neighbors from closest distance to reference state
- average over (sufficiently) many reference states

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mutual false nearest neighbors

equation: unidirectionally coupled Rössler systems and some nonlinear transformation of responder system

new responder variables:

\[
\begin{align*}
    z_1 &= y_1 \\
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\]

measuring interactions

identifying generalized synchronization from time series

mutual false nearest neighbors

example: unidirectionally coupled Rössler systems and some nonlinear transformation of delayed responder system

new responder variables:

\[
\begin{align*}
\dot{z}_1 &= y_1 \\
\dot{z}_2 &= y_2 + 10 \int_{-\infty}^{t} e^{-(t-\tau)} y_1(\tau) d\tau \\
\dot{z}_3 &= y_3
\end{align*}
\]

measuring interactions

identifying generalized synchronization from time series

mutual false nearest neighbors

element: unidirectionally coupled Rössler systems and some nonlinear transformation of delayed responder system

new responder variables:

\[ z_1 = y_1 \]
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mutual false nearest neighbors

$P_{MFNN}$ increases with coupling strength

$\rightarrow$ data-driven estimator for strength of interaction

symmetric ansatz

$\rightarrow$ no indication for direction of interactions

ansatz assumes existence of $\psi$, well-defined properties of $\psi$
(smooth, differentiable, invertible) for strong coupling only!

sensitive to noise, too many parameters

$\rightarrow$ not well suited for time series analysis
measuring interactions
identifying generalized synchronization from time series
nonlinear interdependence

Qs:

why assume existence of $\psi$? (necessary/relevant?)

weak coupling? (is more interesting case)

w.r.t. time series analysis: influence of noise?

existence of (unknown) third system, driving the others?

why assume determinism? (stochastic processes)
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nonlinear interdependence

given observables $x(t)$ and $y(t)$ of systems $X$ and $Y$

appropriate state-space reconstruction (time-delay embedding)

choose reference vectors $x_n = x(t_n)$ and $y_n = y(t_n)$

identify their $k$ nearest neighbors ($\varepsilon$-environment, Euclidean distance) and denote their indices by $r_{n,j}$ and $s_{n,j}$, $j = 1, \ldots, k$

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nonlinear interdependence

uncoupled

strong coupling

reference vectors connected by red lines; neighbors connected by yellow lines

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nonlinear interdependence

define “true” and “false” distances for system X (analogously for Y)

\[
R_n^{(k)}(X) := \frac{1}{k} \sum_{j=1}^{k} (x_n - x_{r_{n,j}})^2 \quad R_n^{(k)}(X|Y) := \frac{1}{k} \sum_{j=1}^{k} (x_n - x_{s_{n,j}})^2
\]

observations:

if X and Y strongly related: \( R_n^{(k)}(X) \sim R_n^{(k)}(X|Y) \)

if X and Y independent: \( R_n^{(k)}(X) \ll R_n^{(k)}(X|Y) \)

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nonlinear interdependence

define “local” and “global” interdependence measure for system $X$ (analogously for $Y$)

$$S^{(k)}_n(X|Y) := \frac{R^{(k)}_n(X)}{R^{(k)}_n(X|Y)}$$

$$S^{(k)}(X|Y) := \frac{1}{N} \sum_{n=1}^{N} S^{(k)}_n(X|Y)$$

- measures confined to unit interval ($S=1$ strong interdependence)
- asymmetry $S(X|Y) \neq S(Y|X)$ reflects different levels of complexity of systems
- no claims about causality $\rightarrow$ active-passive relationship

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nonlinear interdependence

alternatives and extensions

\[ H(\mathbf{X}|\mathbf{Y}) := \frac{1}{N} \sum_{n=1}^{N} \log \frac{R_n^{(k)}(\mathbf{X})}{R_n^{(k)}(\mathbf{X}|\mathbf{Y})} \]

\[ M(\mathbf{X}|\mathbf{Y}) := \frac{1}{N} \sum_{n=1}^{N} \frac{R_n(\mathbf{X}) - R_n^{(k)}(\mathbf{X}|\mathbf{Y})}{R_n(\mathbf{X}) - R_n^{(k)}(\mathbf{X})} \]

\[ L(\mathbf{X}|\mathbf{Y}) := \frac{1}{N} \sum_{n=1}^{N} \frac{G_n(\mathbf{X}) - G_n^{(k)}(\mathbf{X}|\mathbf{Y})}{G_n(\mathbf{X}) - G_n^{(k)}(\mathbf{X})} \]

with the mean and minimal mean rank

\[ G_n(\mathbf{X}) = \frac{N}{2} \] and \[ G_n^{(k)}(\mathbf{X}) = \frac{k+1}{2} \]

measuring interactions

identifying generalized synchronization from time series

nonlinear interdependence

symmetric and antisymmetric estimators for strength and direction of interaction:

\[
A_+^{(k)}(X|Y) = A_+^{(k)}(Y|X) := \frac{1}{2} \left( A^{(k)}(X|Y) + A^{(k)}(Y|X) \right)
\]

\[
A_-^{(k)}(X|Y) = -A_-^{(k)}(Y|X) := \frac{1}{2} \left( A^{(k)}(X|Y) - A^{(k)}(Y|X) \right)
\]
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nonlinear interdependence

comparison with phase-based estimators

diffusively coupled Rössler oscillators

*identical* eigen-frequencies

50 realizations, 4096 data points

embedding dimension 5

embedding delay 1

\[
\begin{align*}
\dot{x}_{1,2} &= -\Omega_{1,2} y_{1,2} - z_{1,2} + c(x_{2,1} - x_{1,2}) \\
\dot{y}_{1,2} &= \Omega_{1,2} x_{1,2} + 0.165 y_{1,2} \\
\dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10) \\
\Omega_1 &= \Omega_2 = 0.89
\end{align*}
\]

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identifying generalized synchronization from time series
nonlinear interdependence

comparison with phase-based estimators

diffusively coupled Rössler oscillators
different eigen-frequencies
50 realizations, 4096 data points
embedding dimension 7
embedding delays (1, 30)

\[
\begin{align*}
\dot{x}_{1,2} &= -\Omega_{1,2} y_{1,2} - z_{1,2} + c(x_{2,1} - x_{1,2}) \\
\dot{y}_{1,2} &= \Omega_{1,2} x_{1,2} + 0.165 y_{1,2} \\
\dot{z}_{1,2} &= 0.2 + z_{1,2}(x_{1,2} - 10) \\
\Omega_1 &= 0.89 \quad \Omega_2 = 0.95
\end{align*}
\]

measuring interactions

identifying generalized synchronization from time series nonlinear interdependence

comparison with phase-based estimators

diffusively coupled Rössler oscillators different eigen-frequencies 50 realizations, 4096 data points embedding dimension 7 embedding delays (1, 30)

impact of frequency mismatch

phase-based: the fast system appears to drive the other system

state-space-based: appears unaffected

measuring interactions

identifying generalized synchronization from time series

nonlinear interdependence

comparison with phase-based estimators

diffusively coupled Rössler (driver) – Lorenz (responder) oscillators
50 realizations, 16384 data points
embedding dimension 7
embedding delay 1

coupling strength \( c \)

\[
\begin{align*}
\dot{x}^R &= -\Omega^R (y^R - z^R) \\
\dot{y}^R &= \Omega^R (x^R + 0.2y^R) \\
\dot{z}^R &= \Omega^R (0.2 + z^R (x^R - 5.7)) \\
\dot{x}^L &= 10(y^L - x^L) \\
\dot{y}^L &= \Omega^L x^L - y^L - x^L z^L + c(y^R)^2 \\
\dot{z}^L &= x^L y^L - \frac{8}{3} z^L
\end{align*}
\]

correct identification?

almost independent on \( c \)?

no identification

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state-space

identifying generalized synchronization from time series

nonlinear interdependence

improved estimators ($M$ and $L$)

diffusively coupled Lorenz oscillators

1000 realizations

embedding dimension +

embedding delay optimal Theiler correction

different noise contaminations

left: X only

middle: X and Y

right: Y only (colors)

correct identification: $\Delta A > 0$

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nonlinear interdependence

what can go wrong?

inappropriate normalization (translation, rotation, …)
all issues related to embedding
strongly coupled systems (direction of interaction)
not controlling for strong correlations in data
(see Theiler correction)

not accounting for different “complexities”
(eigen-frequencies, number of degrees of freedom, noise, …)
measuring interactions

**state-space**

**strength and direction of interaction**

state-space-based estimators

advantages

- more general concept
- can capture all forms of synchronization
- high / moderate robustness against noise

disadvantages

- require appropriate choice of algorithmic parameter
- “faster” system \(\rightarrow\) driver
  (need reliable surrogate test for directionality)
- may be fooled by (unobserved) third system
measuring interactions  

**strength of interaction**  

comparison with other approaches

**test bed**

coupled Hénon systems, Rössler oscillators, and Lorenz oscillators

optimally chosen algorithmic parameters

4096 data points

surrogate correction whenever necessary

**measures:**

linear cross correlation

mutual information

phase-based approaches (Hilbert transform, wavelet transform)

nonlinear interdependency

**event synchronization**  

measuring interactions
strength of interaction
comparison with other approaches

coupled Hénon systems

recall: generalized synchronization, iff $\lambda_i^{(R)} < 0 \ (\forall i)$

measuring interactions

**strength of interaction**

comparison with other approaches

coupled Rössler oscillators

recall: generalized synchronization, iff $\lambda_i^{(R)} < 0 \ (\forall i)$

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Interactions

measuring interactions

state-space

strength of interaction

comparison with other approaches

coupled Lorenz oscillators

recall: generalized synchronization, iff $\lambda_{i}^{(R)} < 0$ ($\forall i$)

measuring interactions

strength of interaction

state-space

correlation between approaches

measuring interactions  

*strength of interaction*  

state-space  

comparison with other approaches  

there is no “best approach”  

dependent on specific application  

choose approach according to quality and type of data  

even combinations of approaches might be useful  

measuring interactions

direction of interaction

test bed

coupled AR models, Hénon systems, Rössler-Lorenz oscillators, Rössler oscillators (large eigen-frequency mismatch), fishery model, two uncoupled systems driven by a third one

optimally chosen algorithmic parameters; 20,000 data points

surrogate correction whenever necessary

measures:

Granger causality (various approaches)
transfer entropy
nonlinear interdependency
measuring interactions state-space

direction of interaction comparison with other approaches

- results of different methods often contradicted each other
- methods differed considerably in their capability to reveal presence and direction of coupling and to distinguish causality from correlation
- outputs of methods difficult to compare
- low specificity was the problem of most methods
- choose the right method for a particular type of data
  “simple” cases → linear methods
  “complex” cases → information-theoretic and/or nonlinear methods

- blind application of any causality test easily leads to incorrect conclusions

Krakovská, A., Jakubík, J., Chvosteková, M., Coufal, D., Jajcay, N. Paluš, M., Phys. Rev. E, 97, 042207, 2018
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measuring interactions

state-space

other ideas (for the sake of completeness)

from (auto) correlation sum

\[ C^{XX}(\epsilon) := \frac{1}{N} \sum_i \left( \frac{1}{N} \sum_j \Theta(\epsilon - |x_i - x_j|) \right) \quad \text{with } C^{XX}(\epsilon) \propto \epsilon^{D_2} \]

to cross-correlation sum

\[ C^{XY}(\epsilon) := \frac{1}{N} \sum_i \left( \frac{1}{N} \sum_j \Theta(\epsilon - |x_i - y_j|) \right) \quad \text{with } C^{XY}(\epsilon) \propto \epsilon^{\min(\dim(X \oplus Y))} \]

measuring interactions

cross-correlation sum

identical systems if:

\[ C^{XX}(\epsilon) = C^{YY}(\epsilon) = C^{XY}(\epsilon) \quad \forall \epsilon \]

need appropriate definition for "similarity"

no information about dynamics

not well suited for time series analysis

measuring interactions

- dimension of interaction dynamics
- Hausdorff distance between attractors
- similarity of state-space densities ($\chi^2$ test)
- similarity based on cross-predictability

→ no or only restricted information about dynamics
→ robustness, computational issues
→ do not provide information about strength and direction of interaction