system theory of imaging systems

**system theory:**
provides mathematical tools to allow transformation of a physically encoded information into another representation without loss of Information (e.g. from position-space to Fourier space)

**transmission system:**

examples:

1D encoded information:
input = language; system = telephone; output = acoustic signal (phone)
information: time-variant membrane pressure

2D encoded information:
input = image; system = xerox machine; output = copy of image
information: location-dependent grey level distribution
Transmission system = imaging system

\[ f(x,y) \rightarrow \text{imaging system} \rightarrow g(x,y) \]

- **Input**: \( f(x,y) \)
- **System**:
  - x-ray dose \( D(x,y) \)
  - attenuation coefficient \( \mu(x,y) \)
  - proton density \( \rho(x,y) \)
- **Output**:
  - x-ray system with amplification film
  - CT-system
  - MRI-system
- **Output Values**:
  - Film
  - Digitized image
  - Image on monitor
**system theory of imaging systems**

**definitions**

an imaging system with:

\[ f_i(x,y) \rightarrow \text{System} \rightarrow g_i(x,y) \]

is called **linear**, iff:

\[ \sum c_i f_i(x,y) \rightarrow \text{System} \rightarrow \sum c_i g_i(x,y) \]

---

**system properties**

an imaging system with:

\[ f(x,y) \rightarrow \text{System} \rightarrow g(x,y) \]

is called **translation-invariant**, iff:

\[ f(x-x_0, y-y_0) \rightarrow \text{System} \rightarrow g(x-x_0, y-y_0) \]
system theory of imaging systems

definitions

mathematical methods for system characterization:

- Dirac function
- Fourier transform and convolution theorem

Time domain:
- impulse response / transfer function

Spatial domain:
- point spread function / modulation transfer function
- auto-/cross-correlation function

Additional aspects for real systems:
- noise
- sampling; aliasing
- filtering
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**definitions**

with rectangular function \( \text{rect}(t) = \begin{cases} 1 \text{ für } |t| \leq 1/2 \\ 0 \text{ für } |t| > 1/2 \end{cases} \)

follows the definition of **δ-function** (Dirac function):

\[
\delta(t) := \lim_{T \to 0} \frac{1}{T} \cdot \text{rect} \left( \frac{t}{T} \right) = \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}
\]

δ-function: infinitely short pulse with infinitely large amplitude
approximation of some function \( f(t) \) with sequence of rect-functions
- approximation of some function $f(t)$ with sequence of rect-functions
- the smaller $T$, the more accurate the approximation
- for $T \to 0$:

$$n \cdot T \to \tau, T \to d\tau, \lim_{T \to 0} s(t) = \delta(t)$$

$$\lim_{T \to 0} s(t) = f(t) = \int_{-\infty}^{+\infty} f(\tau)\delta(t-\tau)d\tau$$
System theory of imaging systems

Definitions

1D Dirac function

Approximation of some function $f(t)$ with sequence of rect-functions

The integral

$$f(t) = \int_{-\infty}^{+\infty} f(\tau) \delta(t - \tau) d\tau$$

is called convolution of function $f$ with Dirac function and can be written as:

$$f(t) \ast \delta(t) = f(t)$$
analogue definitions for 2D case

\[
f(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(p, q) \delta(x - p, y - q) \, dp \, dq
\]

\[
= f(x, y) \ast \delta(x, y)
\]

\(\delta(x,y)\) is a two-dimensional impulse
**definitions**

properties of the $\delta$-function:

- **filtering:**
  \[ f(x_0) = \int_{-\infty}^{+\infty} f(x) \delta(x - x_0) \, dx \]

- **linearity:**
  \[ c_1 \cdot \delta(x) + c_2 \cdot \delta(x) = (c_1 + c_2) \cdot \delta(x) \]

- **symmetry:**
  \[ \delta(-x) = \delta(x) \]

- **elongation:**
  \[ \delta(bx) = \frac{1}{|b|} \cdot \delta(x) \]
system theory of imaging systems

definitions

properties of convolution algebra:

convolution
\[ g(x) = f(x) * h(x) = \int_{-\infty}^{+\infty} f(y)h(x-y)\,dy \]

identity
\[ f(x) = f(x) * \delta(x) = \delta(x) * f(x) \]

commutative-
\[ f(x) * h(x) = h(x) * f(x) \]

associative-
\[ [f(x) * g(x)] * h(x) = f(x) * [g(x) * h(x)] \]

distributive-
\[ f(x) * [c_1 h_1(x) + c_2 h_2(x)] = c_1 f(x) * h_1(x) + c_2 f(x) * h_2(x) \]

linearity
\[ (f(x) * h(x))' = f'(x) * h(x) = f(x) * h'(x) \]

convolution
- in signal processing $h(t)$ is called **impulse response**

- for $h(t) = \delta(t)$, the system is called **ideally distortion-free** since $f(t) = \delta(t) * f(t)$ holds
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definitions

- consider input functions whose amplitudes are influenced by the system, but there are no other changes of form

- such functions are called eigenfunctions

- example: harmonic functions with constant frequency $\omega$.

\[ f(t) = e^{j2\pi\omega t} = \cos(2\pi\omega t) + j\sin(2\pi\omega t) \]
**System theory of imaging systems**

**Definitions**

- **Input**: $f(t)$
- **System**: $h(t)$
- **Output**: $H \cdot f(t)$

**Fourier transform**

- System response to harmonic function at input:

$$f(t) * h(t) = \int h(\tau) \cdot e^{j2\pi\omega(t-\tau)} \, d\tau$$

$$= e^{j2\pi\omega t} \cdot \int h(\tau) \cdot e^{-j2\pi\omega \tau} \, d\tau = H \cdot f(t)$$
- the, in general, complex-valued factor $H$ depends on system, frequency, and input function:

$$H(\omega) = \int h(t) \cdot e^{-j2\pi\omega t} \, dt$$

- $H(\omega)$ is called transfer function (filter response, frequency response).

- since

$$h(t) = \int H(\omega) \cdot e^{j2\pi\omega t} \, d\omega$$

both impulse response $h(t)$ and transfer function $H(\omega)$ are equivalent descriptors of a linear stationary systems
- let \( f(t) \) be a **superposition of harmonic functions**.

- the transformation (from time to frequency domain)

\[
f(t) = \int F(\omega) \cdot e^{j2\pi\omega t} d\omega
\]

and the inverse transformation (from frequency to time domain)

\[
F(\omega) = \int f(t) \cdot e^{-j2\pi\omega t} dt
\]

is called **Fourier transform**
definitions

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variant forms of spelling:

\[ f(t) \longrightarrow F(\omega) \]

\[ f(t) \longrightarrow \bullet F(\omega) \]

\[ F(\omega) = FT (f(t)) \]

\[ f(t) = FT^{-1} (F(\omega)) \]
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definitions

properties of the Fourier transform (I):

- **linearity**
  \[ c_1 \cdot f_1(t) + c_2 \cdot f_2(t) \iff c_1 \cdot F_1(\omega) + c_2 \cdot F_2(\omega) \]

- **time shift**
  \[ f(t - t_0) \iff F(\omega) \cdot e^{-j2\pi\omega t_0} \]

- **time/frequency scaling**
  \[ f(a \cdot t) \iff \frac{1}{|a|} \cdot F\left(\frac{\omega}{a}\right) \]

- **complex conjugate signal**
  \[ f^*(t) \iff F^*(-\omega) \]

- **time reversal**
  \[ f(-t) \iff F(-\omega) \]

- **symmetry**
  \[ F(t) \iff f(\omega) \]
properties of the Fourier transform (II):

- **convolution**
  \[ f_1(t) \ast f_2(t) \Leftrightarrow F_1(\omega) \cdot F_2(\omega) \]

- **multiplication**
  \[ f_1(t) \cdot f_2(t) \Leftrightarrow F_1(\omega) \ast F_2(\omega) \]

- **cross-correlation**
  \[ f_1(t) \otimes f_2(t) \Leftrightarrow F_1^*(\omega) \cdot F_2(\omega) \]

- **auto-correlation**
  \[ f(t) \otimes f(t) \Leftrightarrow |F(\omega)|^2 \]

- **integration**
  \[ \int_{-\infty}^{+\infty} F(\tau) d\tau \Leftrightarrow (j2\pi\omega)^{-1} \cdot F(\omega) + \frac{1}{2} F(0)\delta(\omega) \]

- **differentiation**
  \[ \frac{d^n}{dt^n} f(t) \Leftrightarrow (j2\pi\omega)^n \cdot F(\omega) \]

- **energy/variance (Parseval’s theorem)**
  \[ \int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F(\omega)|^2 d\omega \]
with system properties *linearity* and *translation invariance* (stationarity), we have:

- the system response is fully characterized by a single function
  in time domain: impulse response $h(t)$
  in frequency domain: transfer function $H(\omega)$

- equivalent characterization in reciprocal domain (Fourier transform)

$\Rightarrow$

multiplication in given domain $\propto$ convolution in reciprocal domain
system theory of imaging systems

definitions

Fourier transform of time-dependent signals

$$F(\omega) = \int_{-\infty}^{+\infty} f(t) \exp(-j \cdot \omega t) dt$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) \exp(+j \cdot \omega t) d\omega$$

1D Fourier transform

Fourier transform of location-dependent signals

$$F(u) = \int_{-\infty}^{+\infty} f(x) \exp(-j \cdot 2\pi \cdot ux) dx$$

$$f(x) = \int_{-\infty}^{+\infty} F(u) \exp(+j \cdot 2\pi \cdot ux) du$$

mapping from spatial domain to frequency domain

$$F(u) = |F(u)| \cdot \exp(j \cdot \phi(u))$$

$$|F(u)| = \text{amplitude spectrum}$$

$$\phi(u) = \text{phase}$$
system theory of imaging systems

definitions

example: sinusoidal signal in spatial domain

\[ \lambda := \text{wave length in spatial domain} \]
\[ u = \frac{1}{\lambda} := \text{frequency in Fourier domain} \]

1D Fourier transform
system theory of imaging systems

definitions

example: rectangular function in spatial domain

$$f(x) = \text{rect}(x) = \begin{cases} A \text{ if } x \in [0, x_0] \\ 0 \text{ else} \end{cases}$$
system theory of imaging systems

definitions

example: rectangular function in spatial domain

\[ F(u) = \int_{-\infty}^{+\infty} f(x) e^{-j2\pi u x} \, dx = \int_{0}^{x_0} A e^{-j2\pi u x} \, dx \]

\[ = -\frac{A}{j2\pi u} \left[ e^{-j2\pi u x_0} \right]_{0}^{x_0} = -\frac{A}{j2\pi u} \left[ e^{-j2\pi u x_0} - 1 \right] \]

\[ = \frac{A}{j2\pi u} \left[ e^{j\pi u x_0} - e^{-j\pi u x_0} \right] e^{-j\pi u x_0} = \frac{A}{\pi u} \sin(\pi u x_0) e^{-j\pi u x_0} \]

\[ \Rightarrow \]

\[ |F(u)| = \frac{A}{\pi u} \left| \sin(\pi u x_0) e^{-j\pi u x_0} \right| = A x_0 \left| \frac{\sin(\pi u x_0)}{\pi u x_0} \right| \]
system theory of imaging systems

definitions

1D Fourier transform

example: rectangular function in spatial domain

amplitude spectrum of rect-function in spatial domain

\[ |F(u)| = Ax_0 \left| \frac{\sin(\pi ux_0)}{\pi ux_0} \right| \]

- shifting \( f(x) \) in spatial domain does not affect \( F(u) \)
- only phase of \( F(u) \) is shifted!
system theory of imaging systems

definitions

eexample: image = matrix consisting of digital grey-values

\[
\text{image: } \{\tilde{f}(x_0), \tilde{f}(x_0 + \Delta x), \ldots, \tilde{f}(x_0 + (N - 1) \cdot \Delta x)\}
\]

\[
\text{digitized image: } \{f(0), f(1), \ldots, f(N - 1)\} = f(x); \quad x = 0 \ldots N - 1
\]

digital Fourier transform

\[
F(u) = \frac{1}{N} \cdot \sum_{x=0}^{N-1} f(x) \cdot \exp(-j \cdot 2\pi \cdot ux / N)
\]

\[
f(x) = \sum_{u=0}^{N-1} F(u) \cdot \exp(+j \cdot 2\pi \cdot ux / N)
\]

digital Fourier transformed:

\[
\{F(0), F(1), \ldots, F(N - 1)\} = F(u)
\]

“true” Fourier transformed:

\[
\{\tilde{F}(0), \tilde{F}(\Delta u), \ldots, \tilde{F}((N - 1) \cdot \Delta u)\}
\]

\[
\Delta u = \frac{1}{N \cdot \Delta x}
\]
definitions

1D-FT and convolution theorem

\[ h(t) = f(t) * g(t) = \int_{-\infty}^{+\infty} f(\tau) g(t - \tau) d\tau \]

\[ \text{FT}(h(t)) = \text{FT}(f(t)) \cdot \text{FT}(g(t)) \]

\[ \text{FT}^{-1}(f(t) \cdot g(t)) = \text{FT}^{-1}(f(t)) * \text{FT}^{-1}(g(t)) \]

**convolution in time domain corresponds to multiplication in frequency domain**
system theory of imaging systems

definitions
examples:
convolution with: g1: narrow, g2: broad point spread function PSF

1D Fourier transform
**System theory of imaging systems**

**Definitions**

2D Fourier transform

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \exp(-j \cdot 2\pi \cdot (ux + vy)) \, dx \, dy$$

$$f(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v) \exp(+j \cdot 2\pi \cdot (ux + vy)) \, du \, dv$$

$$F(u, v) = |F(u, v)| \cdot \exp(j \cdot \phi(u, v))$$

$F(u, v)$ = amplitude spectrum

$\phi(u, v)$ = phase

Mapping from spatial domain to frequency domain:
2D-FT of quadratic images (N=M=power-of-2 → FFT):

\[
F(u, v) = \frac{1}{N} \sum_{x,y} f(x, y) \cdot \exp(-j \cdot 2\pi (ux + vy) / N)
\]

\[
f(x, y) = \frac{1}{N} \sum_{u,v} F(u, v) \cdot \exp(+j \cdot 2\pi (ux + vy) / N)
\]

if \(f(x,y)\) real-valued, then:

\[
F(u,v) = F(-u,-v)^*
\]

\[
F(u+N,v) = F(u,v)
\]

\[
F(u,v+N) = F(u,v)
\]
system theory of imaging systems

definitions

2D-FT of quadratic images (N=M=power-of-2 \rightarrow FFT):

with Euler’s formula, we have:

\[ \exp(-j2\pi ux/N) = \cos(-2\pi ux/N) + j\sin(-2\pi ux/N) \]

since \cos and \sin \pi-periodic, we find:

\[ F(u+N) = F(u) \]
system theory of imaging systems

definitions

original

amplitude spectrum

2D Fourier transform

![Graph showing amplitude spectrum and 2D Fourier transform](image)
Definition of 2D Fourier Transform

Original Image

Amplitude Spectrum
system theory of imaging systems

definitions

2D Fourier transform

original

amplitude spectrum
system theory of imaging systems

definitions

2D-FT and convolution theorem

\[ h(x, y) = f(x, y) \ast g(x, y) = \int_{-\infty}^{+\infty} f(x', y') g(x - x', y - y') \, dx' \, dy' \]

2DFT\(h(x, y)\) = 2DFT\(f(x, y)\) \cdot 2DFT\(g(x, y)\)

2DFT\(^{-1}\)(\(f(x, y) \cdot g(x, y)\)) = 2DFT\(^{-1}\)(\(f(x, y)\)) \ast 2DFT\(^{-1}\)(\(g(x, y)\))
system theory of imaging systems

definitions

auto- (cross-) correlation function assesses the correlation of some signal with a delayed copy of itself (or of another signal) as a function of delay (time-lag \( \tau \))

\[
C_{vv}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} v(t)v(t+\tau)dt
\]

\[
C_{vv}(-\tau) = C_{vv}(\tau)
\]

\[
C_{vv}(0) \geq |C_{vv}(\tau)| \quad \forall \tau
\]

normalization such that \( C_{vv}(0) = 1 \) for \( \tau = 0 \)
system theory of imaging systems

definitions

1D case: \[ f(x) \otimes g(x) = \int_{-\infty}^{+\infty} f(x') \cdot g(x + x') dx' \]

correlation theorem:

2D case: \[ f(x, y) \otimes g(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x', y') \cdot g(x + x', y + y') dx' dy' \]

correlation theorem:

auto-correlation function:

(Wiener-Khinchin theorem)
system theory of imaging systems

definitions
example:
autocorrelation functions of some signals

periodic signal

stochastic signal

signal with “memory”
system theory of imaging systems

definitions

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definitions

example:
at which shifts has function \( f(x) \) highest similarity with itself?

the auto-correlation function \( f(x) \odot f(x) \) exhibits several maxima
example:

at which shifts has function g(x) highest similarity with function f(x) ?

the cross-correlation function \( f(x) \otimes g(x) \) exhibits two maxima
main theorem of system theory of imaging systems:

for a linear and translation-invariant system, there exists a function \( h(x, y) \), such that:

\[
g(x, y) = f(x, y) * h(x, y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x', y') h(x - x', y - y') dx' dy'
\]

imaging systems: \( h(x, y) \) is called **point spread function** (PSF)

signal processing: \( h(t) \) is called **impulse response function**:

\[
g(t) = f(t) * h(t) = \int_{-\infty}^{+\infty} f(t) h(t - \tau) d\tau
\]
system theory of imaging systems

point spread function

CT-image of wire (phantom measurement)
system theory of imaging systems

main theorem of system theory of imaging systems:

with convolution theorem, we have:

\[ g(x, y) = f(x, y) * h(x, y) \quad G(u, v) = F(u, v) \cdot H(u, v) \]

the function \( H(u,v) \) is called transfer function and is the Fourier transform of the point spread function \( h(x,y) \)

signal processing: \( H(\omega) \) is called filter (of frequency) response
MTF = absolute value of transfer function normalized to 1 at the origin.

of imaging systems

definitions

modulation transfer function

general definition of MTF:

MTF = modulation transfer function

\[ MTF(u,v) = |H(u,v)| \]

and more exactly:

\[ MTF(u,v) = \frac{|H(u,v)|}{|H(0,0)|} \]

MTF = absolute value of transfer function normalized to 1 at the origin.
modulation transfer function for sinusoidal functions

\[ MTF(u,v) = \left( \frac{|H(u,v)|}{|H(0,0)|} \right) \]

amplitude @output

amplitude @input

contrast @output

contrast @input
system theory of imaging systems

modulation transfer function MTF

CT-image of wire

point spread function

modulation transfer function

ideal system
- let $r(x,y)$ denote a noise input image (quantum noise)

- the image does not contain any information: Fourier spectrum is white, there are no correlations between pixels

- let $R(u,v)$ denote the 2D-Fourier transform of $r(x,y)$ (noise amplitude spectrum).

\[
|R(u,v)^2| = \text{NPS}_\text{input} = \text{noise power spectrum}
\]
white noise: 2D autocorrelation function ≠ 0 at origin only

autocorrelation fct

noise @input

amplitude spectrum
system theory of imaging systems

definitions

- let \( r(x,y) \) pass through ideal imaging system
- system is noiseless (does not add noise to output)
- noise @output can be explained by quantum noise only!!
  (i.e., Detective Quantum Efficiency DQE=1)

- for quantum noise, we have: 
  \[
  \text{DQE} = \frac{\text{mean number of detected } \gamma\text{-quanta}}{\text{mean number of incoming } \gamma\text{-quanta}}
  \]

- with Fourier transform and transfer function

  \[
  \text{image @output: } \ r(x,y) * h(x,y) \rightarrow \text{Fourier transform @output: } \ R(u,v) \cdot H(u,v)
  \]

the noise power spectrum @output reads (Wiener spectrum \( W(u,v) \)):

\[
\text{NPS}_{\text{output}} (u,v) = W(u,v) = |R(u,v)|^2 \cdot |H(u,v)|^2 = |R(u,v)|^2 \cdot \text{MTF}(u,v)^2
\]

(DQE = 1!)
system theory of imaging systems

definitions

- noise power spectrum @output:

\[ \text{NPS}_{\text{output}}(u,v) = k \cdot \text{MTF}(u,v)^2 \]

\[ k = \text{proportionality factor} \]
\[ \text{not dependent on noise power spectrum @input} \]

- noise power spectrum @output and squared MTF have same functional form!

- since FT (autocorrelation function) = noise power spectrum
  \[ \Rightarrow \text{autocorrelation function @output and squared MTF have same functional form!} \]
system theory of imaging systems

definitions

noise @input

amplitude spectrum

noise

noise @output

amplitude spectrum

autocorrelation fct

autocorrelation fct
- neighboring pixel in an output image of an imaging system are no longer independent from each other

- a finite MTF truncates higher spatial frequencies

- a band-limited Fourier spectrum is equivalent to stronger correlations in an image

- an imaging system with finite MTF generates correlations in the output image
Detective Quantum Efficiency (DQE): a measure for image quality factor, by which the system deteriorates the signal/noise ratio if only noise@input:

\[
DQE = \frac{(\text{signal/noise ratio})^2 @\text{output}}{(\text{signal/noise ratio})^2 @\text{input}}
\]

- factor, by which the system deteriorates the signal/noise ratio

- if only noise@input:
  factor, by which the system deteriorates noise

- previous assumption: system does not add noise to output
  (ideal system; DQE = 1)
system theory of imaging systems

definitions

- signal output can be estimated from signal input using the transfer function $H(u,v)$:

$$\text{signal}_{\text{output}}(u,v) = \text{signal}_{\text{input}}(u,v) \cdot H(u,v)$$

- with $MTF(u,v) = |H(u,v)|$, we have

$$\text{signal}^2_{\text{output}}(u,v) = G^2 \cdot \text{signal}^2_{\text{input}}(u,v) \cdot MTF^2(u,v)$$

where $G = \text{amplification factor (gain)}$; system-dependent

⇒ generalized DQE

$$\text{DQE}(u,v) = G^2 \cdot MTF^2(u,v) \cdot \frac{\text{NPS}_{\text{input}}(u,v)}{\text{NPS}_{\text{output}}(u,v)}$$
DQE and MTF for an imaging system

with: \( \text{DQE}(u,v) = G^2 \cdot \text{MTF}^2(u,v) \cdot \frac{\text{NPS}_{\text{input}}(u,v)}{\text{NPS}_{\text{output}}(u,v)} \)

we have: \( \text{NPS}_{\text{output}}(u,v) = G^2 \cdot \frac{\text{MTF}^2(u,v)}{\text{DQE}(u,v)} \cdot \text{NPS}_{\text{input}}(u,v) \)

even if \( \text{DQE}(u,v) = \text{const.} \) for high spatial frequencies \((u,v)\),
the noise power spectrum is reduced by the MTF\((u,v)\)

if \( \text{DQE}(u,v) \) decreases more rapidly to 0 than MTF\((u,v)\),
the noise power spectrum in this frequency band is strongly enhanced
DQE and MTF for an imaging system

- MTF is always finite in real imaging systems
  band limitation, correlations
  diminished spatial resolution

- there is always noise in real imaging systems
  DQE < 1

improving DQE results in a deteriorated MTF and vice versa!
definitions

DQE and MTF for an imaging system
digitization:
conversion of continuous (amplitudes) grey-scale values into
digital discrete (amplitudes) grey-level values

quantization:
conversion of an analogue (signal) image into discrete (values) pixel

quantization error:
example: 10 bit ADC
range: 0 - 1024
single value: \( q = \frac{1024}{2^{10}} = 1 \)
quantization error (q/2) = 0.5
given image $f(x,y)$ and sensors with sensitivity curve $S(x,y)$

signal from sensor $(n,m)$:

$$M_{nm} = \int \int f(x,y) \cdot S(x - n \cdot \Delta x, y - m \cdot \Delta y) \, dx \, dy$$

mathematically: $S(x,y)$ is 2D-Dirac function multiplication (convolution) of image with a **comb-like function**, that attains the value 1 in the center of a pixel and the value 0 otherwise
system theory of imaging systems

definitions

text

sampling

text
definitions

signal processing:
sampling interval and Nyquist frequency

\[ v_n = \sum_{n=-\infty}^{\infty} v(t) \delta(t - n\Delta t) \]

\[ v_n = v(n\Delta t); \quad n = \ldots,-3,-2,-1,0,1,2,3,\ldots \]

\( \Delta t \) is called sampling interval

\[ \omega_{Nyquist} \equiv \frac{1}{2\Delta t} \]
Let \( v(t) \) denote a continuous and band-limited function, sampled with sampling interval \( \Delta t \):

\[
V(\omega) = 0 \quad \forall |\omega| > \omega_{\text{Nyquist}}
\]

where \( V(\omega) \) denotes the Fourier spectrum of \( v(t) \).

\( v(t) \) is then fully determined by the sampling values \( v_n \):

\[
v(t) = \Delta t \sum_{n=-\infty}^{+\infty} v_n \frac{\sin(2\pi\omega_{\text{Nyquist}}(t - n\Delta t))}{\pi(t - n\Delta t)} \propto v(t) * \text{sinc} \left( \frac{t}{\Delta t} \right)
\]
system theory of imaging systems

definitions

image processing:
sampling interval and Nyquist frequency

\[ \Delta x \] is called spatial sampling interval

\[ \omega_{Nyquist} \equiv \frac{1}{2\Delta x} \]

\[ \omega_{Nyquist} \] is the highest spatial frequency
system theory of imaging systems

definitions

sampling theorem

\[ W = \omega_{\text{Nyquist}} \]
- aliasing: sampling a **non-band-limited** continuous function

\[ V(\omega) \neq 0 \quad |\omega| > \omega_{Nyquist} \]

- these spectral components are (somehow) convolved to the interval

\[ |\omega| \leq \omega_{Nyquist} \]

**solution:**

(1) bandwidth of signal known *a priori* or limited prior to sampling using some filter

(2) adequate sampling
system theory of imaging systems

definitions

image with aliasing error

original

scanning points

aliasing
definitions
Moiré effect (change of orientation and frequency)

- high-frequent sinusoidal input image
- inadequately sampled image \( \Delta x << \omega_{\text{Nyquist}} \)
- lowpass-filtered image \( F_{\text{cutoff}} = \omega_{\text{Nyquist}} \)
system theory of imaging systems

definitions

inadequate sampling of an ellipse

number of projections

64 128 256 512

number of projections

64 128 256 512

number of projections

64 128 256 512

number of projections

64 128 256 512

number of projections

64 128 256 512
function in (b) not band-limited:
strong edges = Dirac functions = *white* Fourier spectrum
requires *tapering* prior to digitization
(multiplication with suitable “window function” (other than boxcar))
sampling: multiplication of signal (image) with comb-like function

periodisation: convolution of signal with comb-like function
(Nyquist condition: periodic continuation)

sampling theorem: when following the Nyquist condition, the band-limited interpolation (sinc-series) yields the original function, or leads to aliasing errors otherwise

sampling frequency: at least twice as high as the signal's maximum frequency.
let $f_G$ denote the cut-off frequency

high-pass filter:
- all frequency components smaller than $f_G$ are set to 0 (delete)
- all frequency components larger than $f_G$ are multiplied by 1 (allowed to pass through)

low-pass filter:
- all frequency components larger than $f_G$ are set to 0 (delete)
- all frequency components smaller than $f_G$ are multiplied by 1 (allowed to pass through)
system theory of imaging systems

definitions

- **low-pass**
  - $S(f)$
  - $f_G$ to $f$
  - pass band
  - stop band

- **high-pass**
  - $S(f)$
  - $f_G$ to $f$

- **band-pass**
  - $S(f)$
  - $f$

- **band-stop (notch)**
  - $S(f)$
  - $f$

filtering
system theory of imaging systems

definitions

original

low-pass filtered

filtering

MTF(u,v) of low-pass filter
system theory of imaging systems

definitions

original

high-pass filtered

filtering

MTF(u,v) of high-pass filter
system theory of imaging systems

definitions

original

low-pass filtered

high-pass filtered

“smearing”

“edge enhancement”
system theory of imaging systems

definitions

- synthetic checkerboard 120x120 grey levels
- grey level profile along one line

Original

White noise (std. dev. = 5)

“Salt and pepper” noise

filtering
**mean filter**

smoothing through local averaging (low-pass)

assumption:
- image has low-frequency content only
- noise has high-frequency content only

spatial domain:
- rectangular kernel
  - replace original pixel with weighted sum of neighboring pixel

**caveat:** produces echoes (“ringing”) due to convolution with $\sin(x)/x$
system theory of imaging systems

mean filter (3x3 low-pass)

original image + white noise

filtered image
system theory of imaging systems

mean filter (3x3 low-pass)

original image + salt and pepper noise

filtered image
Gauss filter
extension of mean filter
replace rectangular with Gauss function

diminishes echoes
more advantageous than mean filter
(FT of Gauss function is Gauss function)
easy-to-implement; fast:
Gauss kernel can be decomposed:
2D-filter can be realized by two 1D-filter
Gauss filter (kernel width: 5 pixel)
system theory of imaging systems

Gauss filter

no filter

kernel width: 10 pixel

kernel width: 20 pixel
median is the value separating the upper half from the lower half of a data sample

for discrete data:
- sort by size (rank order)
- then:

\[
\hat{x} = \frac{x_N + x_{N+1}}{2} \quad \text{for } N \text{ even}
\]

\[
\hat{x} = x_{N+1} \quad \text{for } N \text{ odd}
\]

- use median (or central value) if data sample not normally distributed
- median is insensitive to outlier
median filter (rank order filter)

for each Pixel \( p(i,j) \) and its \( n \times n \) neighborhood,
- sort pixel values by size
- replace \( p(i,j) \) with median

pros:
diminishes echoes
retains discontinuities
no influence of very noisy pixel

cons:
longer calculation times
system theory of imaging systems

median filter

white noise
(\(\sigma = 2\))

median filter
2 x 2 neighborhood
system theory of imaging systems

median filter

salt and pepper noise

median filter with 2x2 neighborhood
system theory of imaging systems

median filter
system theory of imaging systems

summary (I)

prerequisite: imaging system is linear and translation invariant

(1) the mapping of some physical observable \( f(x,y) \) using an imaging system can be described mathematically as a convolution of the function \( f(x,y) \) with a function \( h(x,y) \) that fully characterizes the imaging system. We have: \( g(x,y) = f(x,y) \ast h(x,y) \), where \( g(x,y) \) denotes the output of the imaging system.

(2) the convolution theorem allows an equivalent description of the system in a reciprocal space (convolution in spatial domain corresponds to multiplication in Fourier space). We have: \( G(u,v) = F(u,v) \times H(u,v) \), where \( G, F, H \) denote the Fourier transforms of the functions \( f, g, h \).

(3) in the spatial domain, \( h(x,y) \) is called point spread function, the function \( H(u,v) \) in the reciprocal domain is called transfer function.
summary (II)

(4) important characterizing measures for an imaging system (in terms of …):

… spatial resolution
- point spread function in spatial domain (sampling)
- modulation transfer function \( \text{MTF}(u,v) = |H(u,v)| \) in Fourier domain

… noise
- **Detective Quantum Efficiency** \( \text{DQE}(u,v) = \frac{\text{SNR}_{\text{input}}}{\text{SNR}_{\text{output}}} \) (SNR = Signal-to-Noise Ratio)

(5) for an ideal system:
- point spread function = Dirac function (distortion-free system)
- MTF = 1 for all spatial frequencies
- DQE = 1 for all spatial frequencies

for a real system:
- limited spatial resolution
- point spread function non-Dirac-like (broad)
- MTF decreases with higher spatial frequencies (artificial correlations in image)
- DQE < 1 (quantum noise, Poisson distribution)
(6) improving the signal-noise ratio (DQE) results in a diminished resolution (MTF) and vice versa.

(7) acquisition: obey the sampling (Nyquist) theorem to avoid aliasing errors: sampling rate must be chosen at least twice the maximum spatial frequency in the object of interest

(8) tapering, if object not band-limited (minimization of broad-band artificial contributions to image)

(9) post-processing of image: - noise contaminations can be minimized by filtering the image - choose filter appropriately! (the best filtering is no filtering)