Lyapunov Exponents

Stability and Predictability

from Time Series

Long-term Behavior and Stability

long-term:

$t \rightarrow \infty$ (can not be achieved when observing real systems)

observation time:

$T << \infty$

largest characteristic time scale of system

$t_c < T$

Long-term Behavior and Stability

different types of long-term behavior:

- *unlimited growth*
  in practice: can usually not be observed
  in model studies: temporal stabilization or change of model

- *bounded dynamics*
  fixed point, equilibrium
  periodic, quasi-periodic motion
  chaotic motion

Q: how stable is the dynamics,
when perturbing the system?
(when changing control parameters? mostly not considered)

Long-term Behavior and Stability

stability dogma (Andronov & Pontryagin, 1930s):

“since all mathematical models are simplifications and abstractions, models that are relevant for applications must be structurally stable”

↓ however

simple models that are composed of physically acceptable unit are structurally unstable

↓ (cf. weak/strong causality)

which (initial) states lead to the same / a similar long-term behavior?

→ concept of **Lyapunov-stability**

Long-term Behavior and Stability

A.M. Lyapunov

Consider an, in general, nonlinear dynamical system

$$\dot{\boldsymbol{x}} = f(\boldsymbol{x}(t), \beta), \quad \boldsymbol{x} \in \mathbb{R}^d$$

Suppose f has an equilibrium at $\boldsymbol{x}_e$ so that $f(\boldsymbol{x}_e) = 0$, then this equilibrium

- is *Lyapunov stable*, if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$, such that if $\|\boldsymbol{x}(t=0) - \boldsymbol{x}_e\| < \delta$ then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| < \epsilon$ for every $t \geq 0$,
- is *asymptotically stable*, if it is Lyapunov stable and there exists $\delta > 0$ such that if $\|\boldsymbol{x}(t=0) - \boldsymbol{x}_e\| < \delta$, then $\lim_{t\to\infty} \|\boldsymbol{x}(t) - \boldsymbol{x}_e\| = 0$,
- is *exponentially stable*, if it is asymptotically stable and there exist $\alpha > 0, \gamma > 0, \delta > 0$ such that if $\|\boldsymbol{x}(t=0) - \boldsymbol{x}_e\| < \delta$, then $\|\boldsymbol{x}(t) - \boldsymbol{x}_e\| \leq \alpha \|\boldsymbol{x}(0) - \boldsymbol{x}_e\| e^{-\gamma t}$, for all $t \geq 0$,

where $\|\cdot\|$ denotes, e.g., the Euclidean or the Manhattan norm.

A. M. Lyapunov, The general problem of motion stability, Ann. Math. Stud. 17 (1892).

Long-term Behavior and Stability

The aforementioned notions of *equilibrium stability* can be generalized to *orbital stability* (closed trajectory; i.e., periodic, quasi-periodic, or non-periodic orbit):

A trajectory $\Phi(t)$ is called *Lyapunov-stable* if for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$, such that the trajectory of any solution $\boldsymbol{x}(t)$ starting at the δ-neighborhood of $\Phi(t)$ remains in the ϵ-neighborhood of $\Phi(t)$ for all $t \geq 0$.

Linear stability analysis:

- consider small perturbation (of equilibrium/trajectory)
- expand f in Taylor-series
- check eigenvalues of Jacobian; stable, if all have strictly negative parts
- real part of the largest eigenvalue (**Lyapunov exponent**) determines time to return to equilibrium/trajectory after perturbation

Long-term Behavior and Stability

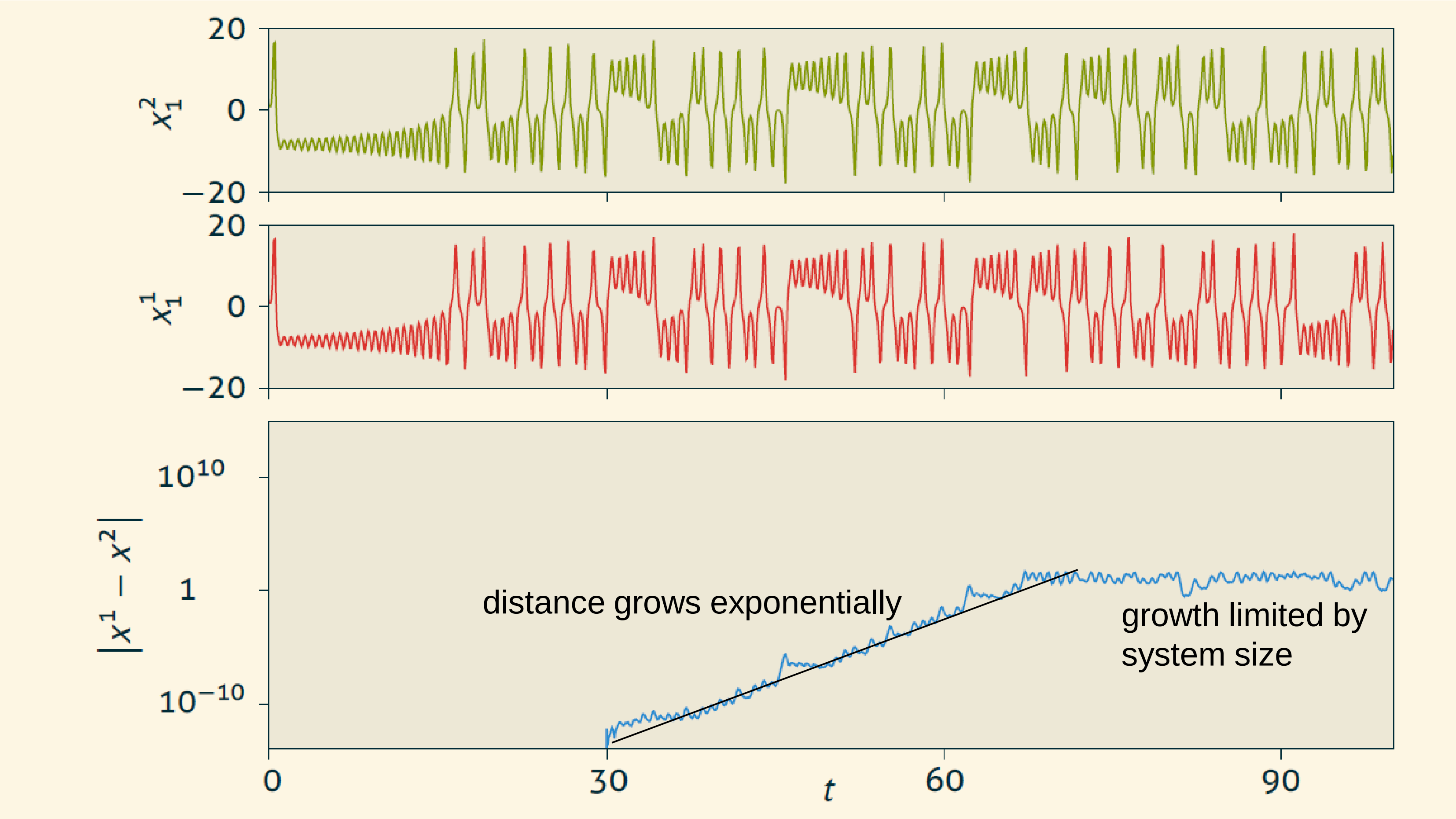
chaotic motion is (locally) Lyapunov-unstable:

divergence:
distance between initially close trajectory segments grows exponentially in time (stretching mechanism)

convergence:
divergence of initially close segments limited by system size; when reached, distance shrinks again (folding mechanism)

Long-term Behavior and Stability

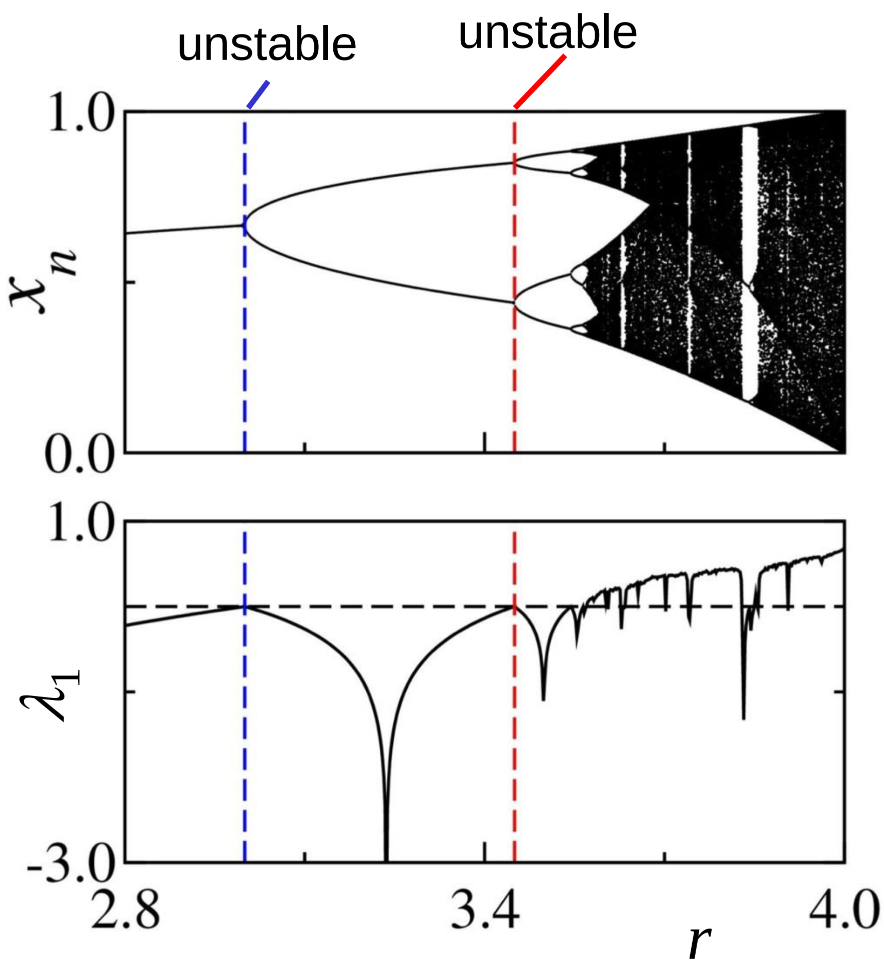
Example: two identical Lorenz oscillators with initial conditions; one oscillator is slightly perturbed (10^{-14}) at t = 30



from: G. Ansmann

Long-term Behavior and Stability

Example: logistic map: $x_{n+1} = r x_n (1 - x_n); \quad x_n \in [0,1]; \quad r \in [0,4]$



λ_1 = largest Lyapunov exponent (dominates the dynamics)

Long-term Behavior and Stability

spectrum of Lyapunov exponents: $\lambda_i \text{ with } i = 1, \ldots, d$

characterize growth rates in different local directions of phase space

Lyapunov exponents and divergence: $\sum_{i=1}^{d} \lambda_i = \nabla \cdot f$

dissipative system: $\sum_{i=1}^{d} \lambda_i < 0$

largest Lyapunov exponent:

$$\lambda_1 = 0 \rightarrow \text{regular dynamics}$$
$$\lambda_1 > 0 \rightarrow \text{chaotic dynamics}$$
$$\lambda_1 < 0 \rightarrow \text{fixed-point dynamics}$$
$$\lambda_1 \rightarrow \infty \rightarrow \text{stochastic dynamics}$$

Lyapunov exponents from time series

model:	***field data:***
continuous trajectories	→ discrete trajectories
actual phase space	→ reconstruction
evolution of arbitrary states	→ available trajectories
equations of motion	→ available trajectories

and of course: finite data, noise, …

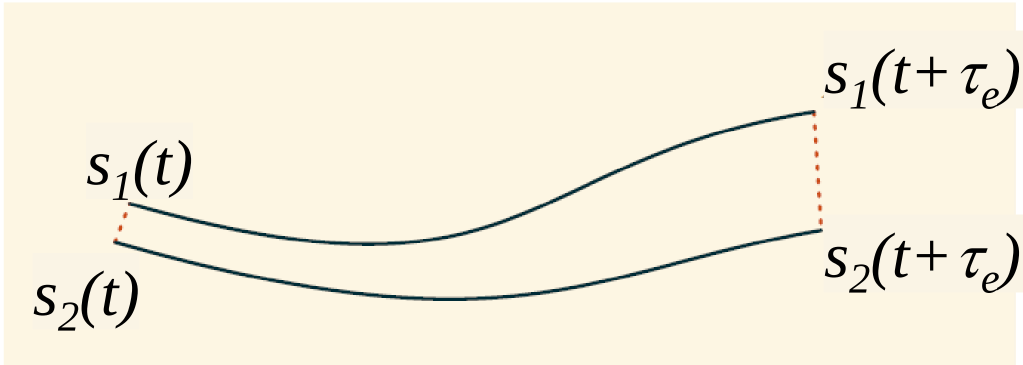
concepts and algorithms (most widely used)***:***

- spectrum of Lyapunov exponents (in general, hard to estimate)
  (Sano & Sawada, 1985; Eckmann et al., 1986; Stoop & Meier, 1988; Stoop & Parisi, 1991)

- largest Lyapunov exponent
  (Wolf et al. 1985; Rosenstein et al., 1993; Kantz, 1994)

Largest Lyapunov exponent — Idea

Consider evolution of two nearby trajectory segments s_1 and s_2



For infinitesimally close trajectory segments $(||s_1(t) - s_2(t)|| \rightarrow 0)$ and for infinite time evolution ($\tau_e \rightarrow \infty$) the distance between segments grows or shrinks exponentially:

$$||s_1(t+\tau_e) - s_2(t+\tau_e)|| = ||s_1(t) - s_2(t)||\, \mathrm{e}^{\lambda_1 \tau_e}$$

Largest Lyapunov exponent Idea

$$||s_1(t+\tau_e) - s_2(t+\tau_e)|| = ||s_1(t) - s_2(t)||\, \mathrm{e}^{\lambda_1 \tau_e}$$

Solve for λ_1 and implement the limits.

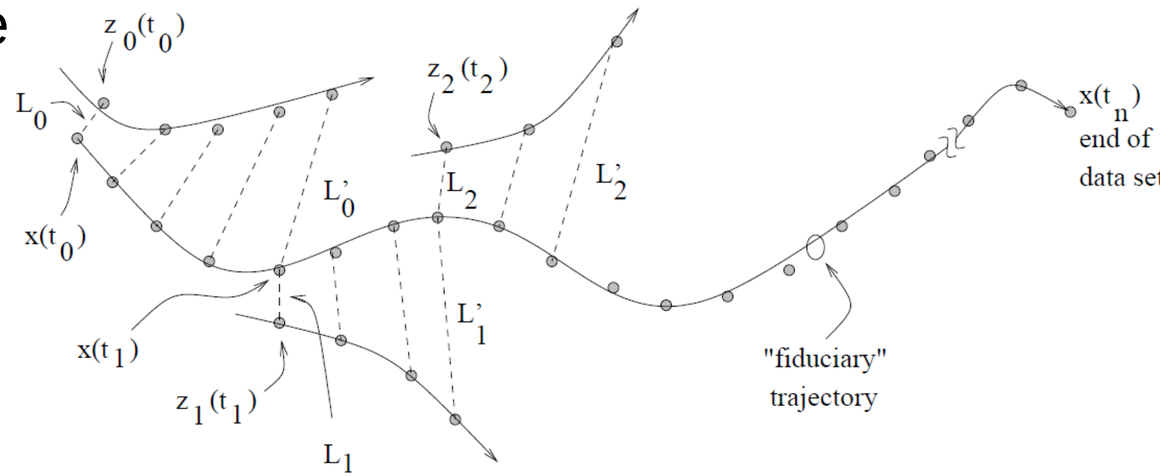
Let s_1 and s_2 denote two near trajectory segments of the dynamics. The first Lyapunov exponent is defined as:

$$\lambda_1 := \lim_{\tau_e \to \infty} \lim_{||s_1(t)-s_2(t)|| \to 0} \frac{1}{\tau_e} \ln \left(\frac{||s_1(t+\tau_e)-s_2(t+\tau_e)||}{||s_1(t)-s_2(t)||} \right)$$

Also: *largest Lyapunov exponent* or just *Lyapunov exponent*.

Largest Lyapunov exponent — Wolf-Algorithm

estimates the dominant Lyapunov exponent from a time series by monitoring orbital divergence



1. reconstruct phase space
2. pick $x(t_0)$ on *fiduciary* trajectory
3. find nearest neighbor $z_0(t_0)$
4. compute $||z_0(t_0) - x(t_0)|| =: L_0$
5. follow *difference trajectory* (dashed line forwards in time and compute $||z_0(t_i) - x(t_i)|| =: L_0(i)$. Increment i until $L_0(i) > \epsilon$, call that value L'_0 and that time t_1
6. find $z_1(t_1)$, the *nearest neighbor* of $x(t_1)$, and loop to step 4. Repeat procedure to the end of *fiduciary* trajectory ($t = t_n$). Keep track of the L_i and L'_i

Find largest (positive) Lyapunov exponent from:

$$\lambda_1 \approx \frac{1}{N\Delta t} \sum_{i}^{M-1} \log_2 \left(\frac{L'_i}{L_i} \right)$$

where M denotes number of loops, and N number of time steps on fiduciary trajectory; $N\Delta t = t_n - t_0$

Wolf, A., Swift, J. B., Swinney, H. L., & Vastano, J. A. (1985). Determining Lyapunov exponents from a time series. Physica D 16(3), 285-317.

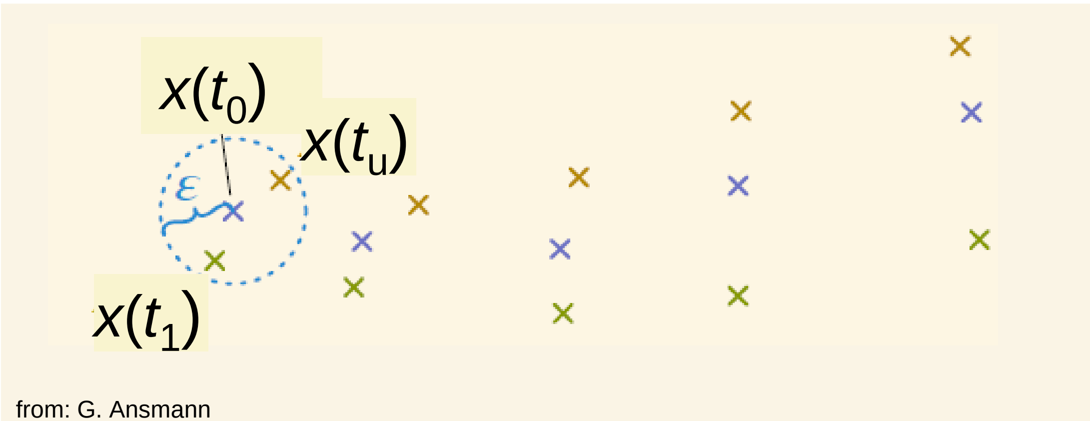
Largest Lyapunov exponent — Wolf-Algorithm

Limitations

- too many parameters that have to be chosen a priori
- problems may be obfuscated:
  no exponential growth due to noise
  embedding dimension *m* too small
- highly sensitive to noise
- difficult to find neighboring trajectory segment with required properties

→ need a different way to ensure alignment to direction of largest growth

Largest Lyapunov exponent Rosenstein-Kantz Algorithm



1. choose reference state $x(t_0)$ and all states $x(t_1)$, …, $x(t_u)$ in ε-neighborhood

2. for given τ_e, define average distance of respective trajectory segments from the initial one as

$$s(t,\tau_e) := \frac{1}{u}\sum_{j=1}^{u} ||x(t_0+\tau_e) - x(t_j+\tau_e)||$$

3. average over all states as reference states:

$$S(\tau_e) := \frac{1}{N}\sum_{t=1}^{N} \ln s(t,\tau_e)$$

4. obtain largest Lyapunov exponent from region of exponential growth of $S(\tau_e)$

Largest Lyapunov exponent Rosenstein-Kantz Algorithm

82 H. Kantz / Physics Letters A 185 (1994) 77–87

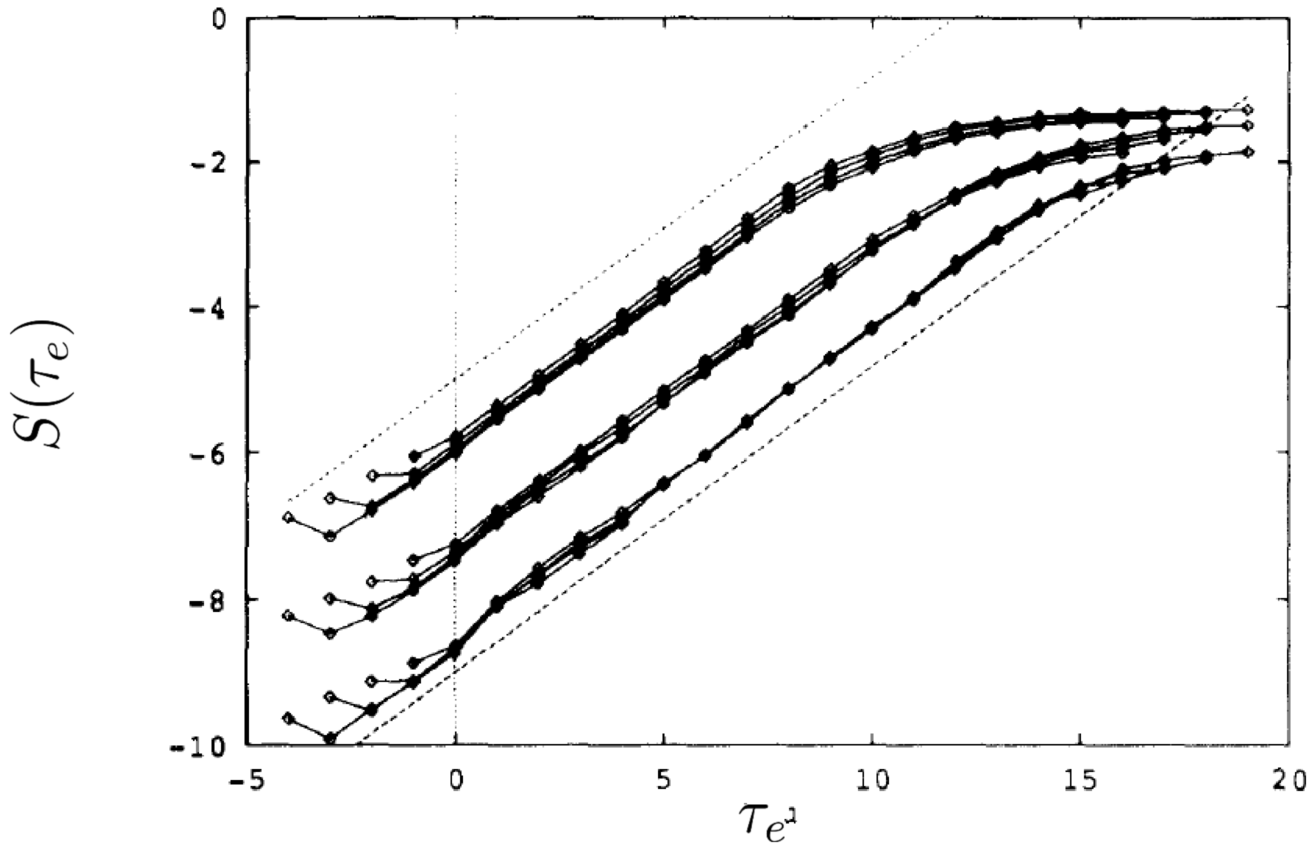


Fig. 2. $S(\tau)$ for a Hénon trajectory of length 2000. The different curves correspond to ϵ = 0.0005, 0.002 and 0.008 (the three bunches from bottom to top) and embedding dimension m = 2–5. The dashed lines have slopes $\lambda_{\text{exact}} = 0.4169$. Again $\tau \leqslant 0$ corresponds to the components used to define the local neighbourhoods.

Largest Lyapunov exponent Rosenstein-Kantz Algorithm

Mind how you average

1. average over the neighborhood of a reference state $\longrightarrow s(t\ ,\tau_{\mathrm{e}})$
2. average $s(t\ ,\tau_{\mathrm{e}})$ over all reference states $\longrightarrow S(\tau_{\mathrm{e}})$
3. Obtain λ_1 from slope of $S(\tau_{\mathrm{e}})$

Density of states in a region of the attractor affects:

- reference states
- states in neighborhood of given reference state

Separating averaging in steps 1 and 2
(instead of averaging of all ε-close pairs)
ensures that density is accounted for only once
(and not twice)

Largest Lyapunov exponent Rosenstein-Kantz Algorithm

Advantages and Problems

- region of exponential growth can be determined a posteriori
  (be careful of wishful thinking though)
- absence of exponential growth usually detectable
  (but only usually)
- region of strong noise influence can be detected and excluded
- can only determine the largest Lyapunov exponent

Largest Lyapunov exponent extensions

- tangent-space methods

→ require estimate of Jacobian

- spectrum of Lyapunov exponents

→ requires a lot of data

Largest Lyapunov exponent units

for flows: **inverse seconds**

for maps: **inverse iterations**

other choices: bits/second or bits/iteration

Spectrum of Lyapunov exponents and type of dynamics

For bounded, continuous-time dynamical systems, we have:

signs of Lyapunov exponents	dynamics
−, −−, −−−, …	fixed point
+, ++, +++, …, +0, ++0, …	not possible (unbounded)
0, 00, 000, …	no dynamics ($f = 0$)
0−, 0−−, 0−−−, …	periodic / limit cycle
00−, 00−−, 00−−−, …	quasiperiodic (torus)
000−, 0000−, …, 000−−, …	quasiperiodic (hypertorus)
+0−, +0−−, +0−−−, …	chaos
++0−, +++0−, …, ++0−−, …	hyperchaos
∞, …	noise

Largest Lyapunov exponent what can go wrong?

field applications

- number of data points ($\lim N \to \infty$)
- data precision
  adopt to requirement of small ε-neighborhood
- strong correlations in data (sampling interval)
  use Theiler correction (see Dimensions)
- noise
  similar impact as with Dimensions
- filtering
  classical filter affect negative Lyapunov exponents only
  due to adding a (passive) system → extra Lyapunov exponent
  magnitude ~ cutoff frequency

Largest Lyapunov exponent what can go wrong?

False indications of chaos:

- unbounded orbits can have $\lambda_1 > 0$

- orbits can separate but not exponentially

(check boundedness and be sure orbit has adequately sampled attractor; check for contraction to zero within machine precision)

- can have **transient chaos***

(double-check with other methods)

*e.g., Y.-C. Lai and T. Tél, Transient Chaos, Complex Dynamics on Finite-Time Scales (Springer, New York, 2011)

Largest Lyapunov exponent transient chaos: an example

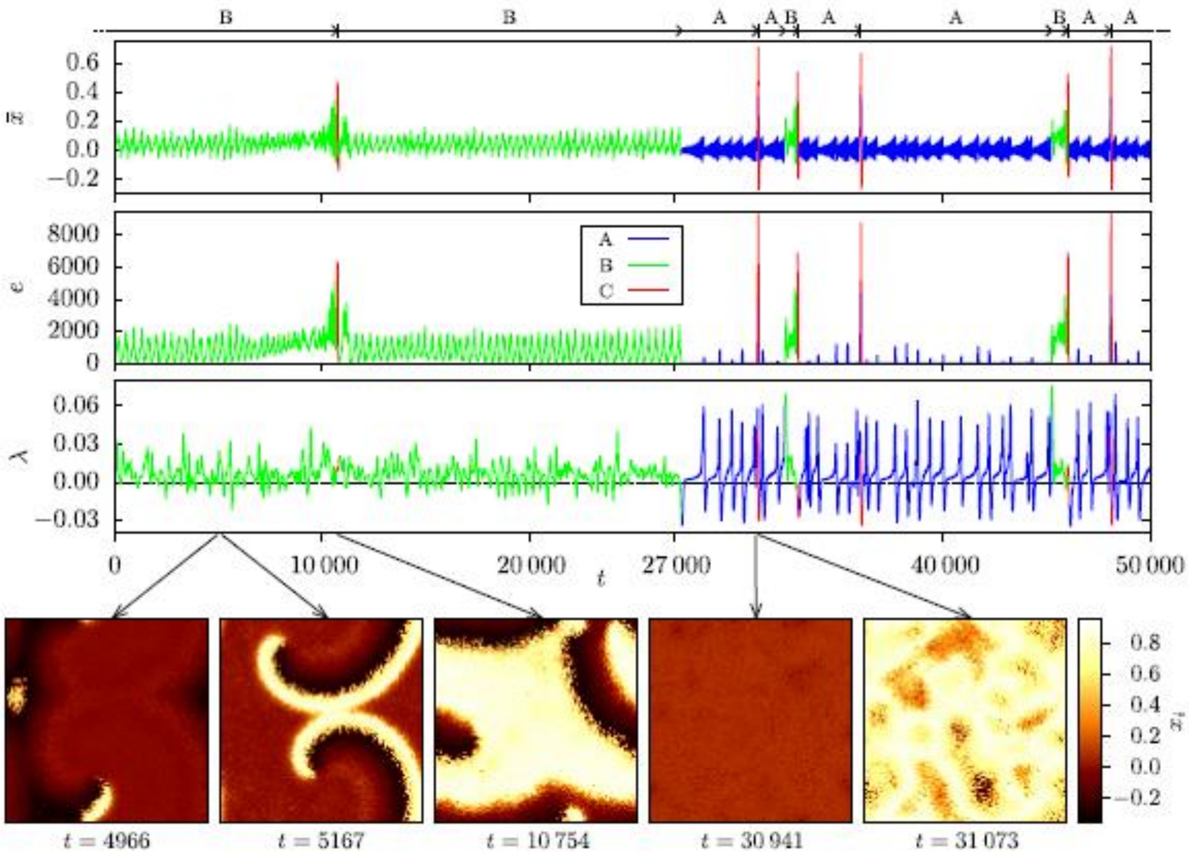


FIG. 1. First to third row: Exemplary temporal evolutions of $\bar{x}$, of the number e of units with $x_i > 0.4$ ("excited units"), and of an estimate λ of the largest local Lyapunov exponent (temporal evolution smoothed with a Gaussian kernel with a width of 30 to improve readability). The line colors indicate the patterns as automatically classified (blue: A, low-amplitude oscillations; green: B, waves; red: C, extreme events). These patterns are also indicated at the very top with pattern C being indicated with a vertical line. Bottom: Snapshots of the spatial distribution of $x_i(t)$ at times corresponding to selected local minima and maxima of $\bar{x}$ (from left to right): adjacent minimum and maximum around $t = 5000$, the maximum during the event around $t = 10\,000$, the minimum before the event around $t = 30\,000$, and the maximum during that event. Units are represented by pixels, which are arranged according to the lattice underlying the small-world network and whose color encodes the value of the respective x_i [65].

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Self-Induced Switchings between Multiple Space-Time Patterns on Complex Networks of Excitable Units

Gerrit Ansmann,[1,*] Klaus Lehnertz,[1,†] and Ulrike Feudel[2,‡]
[1]*Department of Epileptology, University of Bonn, Sigmund-Freud-Straße 25, 53105 Bonn, Germany, Helmholtz Institute for Radiation and Nuclear Physics, University of Bonn, Nussallee 14–16, 53115 Bonn, Germany, and Interdisciplinary Center for Complex Systems, University of Bonn, Brühler Straße 7, 53175 Bonn, Germany*
[2]*Theoretical Physics/Complex Systems, ICBM, Carl-von-Ossietzky University of Oldenburg, Carl-von-Ossietzky-Straße 9–11, Box 2503, 26111 Oldenburg, Germany, and Research Center Neurosensory Science, Carl von Ossietzky University of Oldenburg, Carl-von-Ossietzky-Straße 9–11, 26111 Oldenburg, Germany*
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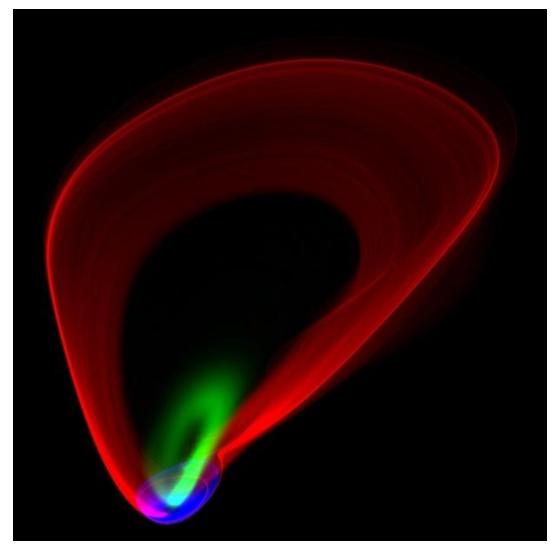
$$\dot{x}_i = x_i(a - x_i)(x_i - 1) - y_i + \frac{k}{m}\sum_{j=1}^{n} A_{ij}(x_j - x_i),$$

$$\dot{y}_i = b_i x_i - c y_i.$$



Largest Lyapunov exponent Interpretation

- stability and type of the dynamics:
  - $\lambda_1 > 0$ chaos, unstable dynamics
  - $\lambda_1 = 0$ regular dynamics
  - $\lambda_1 < 0$ fixed-point dynamics

- quantification of loss of information due to action of nonlinearity

- prediction horizon:

$$T_{\mathrm{p}} \approx \frac{-\ln(\rho)}{\sum_{i,\lambda_i>0} \lambda_i}$$

where:
ρ denotes accuracy of measurement (initial state)
$\sum_{i,\lambda_i>0}$ is sum of positive Lyapunov exponents

Kaplan-Yorke conjecture

relationship between dimension and Lyapunov exponents

$$D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}, \quad \text{where} \;\; \sum_{i=1}^{k} \lambda_i \geq 0 \;\; \text{and} \;\; \sum_{i=1}^{k+1} \lambda_i < 0$$

Kaplan-Yorke dimension D_{KY} equals information dimension D_1
(Note: conjecture not generally valid!)

example:
- Hénon map with parameters a = 1.4 and b = 0.3
- λ_1 = 0.603, λ_2 = -2.34
- we find with k = 1:

$$D_{KY} = k + \frac{\lambda_1}{|\lambda_2|} = 1 + \frac{0.603}{|-2.34|} = 1.26$$

Kaplan-Yorke conjecture

relationship between dimension and Lyapunov exponents